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Persistent URL: http://dml.cz/dmlcz/700291

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A Functional Differential Equation in Banach Spaces

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Abstract. In this paper we prove the existence of pseudo-solution and weak solution for the Cauchy problem $x' = Fx, \ x(0) = x_0, \ t \in [0, a]$.

AMS Subject Classification. 34G20

Keywords. Functional differential equation, existence theorem, weak solution, pseudo-solution

The study of the Cauchy problem for differential and functional differential equations in a Banach space relative to the strong topology has attracted much attention in recent years. However a similar study relative to the weak topology was studied by many authors, for example, Szep [11], Mitchell and Smith [9], Szula [12], Kubiaczyk [6, 7], Kubiaczyk and Szula [8], Cichoń [1], Cichoń and Kubiaczyk [2], and others.

Let $E$ be a Banach space, $E^*$ the dual space. We set $B_b(x_0) = \{ x \in E : \| x - x_0 \| \leq b \}, \ (b > 0)$. We denote by $C(I, E)$ the space of all continuous function from $I$ to $E$, and by $(C(I, E), \omega)$ the space $C(I, E)$ with the weak topology. Put

$$\tilde{B} = \{ x \in C(J, E) : x(J) \subset B_b(x_0), \| x(t) - x(s) \| \leq M|t - s|, \ \text{for} \ t, s \in J \} ,$$

note that $\tilde{B}$ is nonempty, closed, bounded, convex and equicontinuous, where $J = [0, h], \ h = \min\{ a, \frac{b}{M} \}$ and $M > 0$ is a constant.

We deal with the Cauchy problem:

$$x' = Fx, \quad x(0) = x_0, \quad t \in I = [0, a], \quad (1)$$

in the case of $F$ being an bounded operator of Volterra type from $\tilde{B}$ into $P(I, E)$ (the space of all Pettis integrable functions on $I$).

Let us introduce the following definitions.

This is the final form of the paper.
Definition 1. $F$ is said to be of Volterra type if for $x_1, x_2 \in \tilde{B}$ and for any $s_o > 0$ the equality $x_1(t) = x_2(t)$ for $t < s_o$ implies $(Fx_1)(t) = (Fx_2)(t)$ for $t \leq s_o$.

Now fix $x^* \in E^*$, and consider
\[ (x^*x)'(t) = x^*((Fx)(t)), \quad t \in I. \tag{1'} \]

Definition 2. A function $x : I \to E$ is said to be a pseudo-solution of the Cauchy problem (1) if it satisfies the following conditions:

(i) $x(\cdot)$ is absolutely continuous,
(ii) $x(0) = x_o$,
(iii) for each $x^* \in E^*$ there exists a negligible set $A(x^*)$ (i.e., mes $A(x^*) = 0$), such that for each $t \notin A(x^*)$,
\[ x^*(x'(t)) = x^*((Fx)(t)) \, . \]

Here $'$ denotes a pseudoderivative (see Pettis [10]).

In other words, by a pseudo-solution of (1) we will mean an absolutely continuous function $x(\cdot)$, with $x(0) = x_o$, satisfying (1') a.e. for each $x^* \in E^*$.

Definition 3. A function $r : [0, \infty) \to [0, \infty)$ is said to be a Kamke function if it satisfies the following conditions:

(i) $r(0) = 0$,
(ii) $u(t) \equiv 0$ is the unique solution of the integral equation
\[ z(t) = \int_0^t r(z(s))ds \, , \quad t \in I \, . \]

Lemma 4 ([9]). Let $H \subset C(I, E)$ be a family of strongly equicontinuous functions. Then
\[ \beta_c(H) = \sup_{t \in I} \beta(H(t)) = \beta(H(I)) \, , \]
where $\beta_c(H)$ denote the measure of weak noncompactness in $C(I, E)$ and the function $t \to \beta(H(t))$ is continuous.

Now suppose that:

(*) For each strongly absolutely continuous function $x : J : \to E$, $(Fx)(\cdot)$ is Pettis integrable, $F(\cdot)$ is weakly-weakly sequentially continuous, then the existence of a pseudo-solution of (1) is equivalent to the existence of a solution for
\[ x(t) = x_o + \int_0^t (Fx)(s)ds \, , \quad t \leq t_o \, , \quad (2) \]
where the integral is in the sense of Pettis (see [10]).
Theorem 5. Let $F$ be a bounded continuous operator of Volterra type from $\tilde{B}$ into $P(I,E)$ and under the assumption $(\ast)$ and
\[
\beta\left(\bigcup\{(Fx)[J] : x \in \tilde{X}\}\right) \leq r(\beta(\tilde{X})) ,
\] holds for every subset $\tilde{X}$ of $\tilde{B}$, where $r$ is a non-decreasing Kamke function and $\beta$ is the measure of weak noncompactness. Then the set $S$ of all pseudo-solutions of the Cauchy problem (1) on $J$ is non-empty and compact in $(C(J,E),w)$.

Proof. Put
\[
Tu(t) = x_o + \int_0^t Fu(s)ds , \quad t \in I, \quad u \in \tilde{B} ,
\]
where the integral is in the sense of Pettis.

By our assumptions the operator $T$ is well defined and maps $\tilde{B}$ into $\tilde{B}$.

Using Lebesgue’s dominated convergence theorem for the Pettis integral (see [4]), we deduce that $T$ is weakly sequentially continuous.

Suppose that $V = \overline{\text{Conv}}(\{x\} \cup T(V))$ for some $V \subset \tilde{B}$. We will prove that $V$ is relatively weakly compact, thus Theorem 1 in [7] is satisfied.

From the definition of $\tilde{B}$ and Lemma 4 it follows that the function $v : t \rightarrow \beta(V(t))$ is continuous on $J$.

For fixed $t \in J$, divide the interval $[0, t]$ into $m$ parts:
\[
0 = t_0 < t_1 < \cdots < t_m = t , \quad \text{where} \quad t_i = it/m , \quad i = 0, 1, 2, \ldots, m .
\]

Put
\[
V([t_{i-1}, t_i]) = \{u(s) = u \in V, \quad t_{i-1} \leq s \leq t_i\} .
\]

By Lemma 4 and the continuity of $v$ there is $s_i \in [t_{i-1}, t_i]$ such that
\[
\beta(V([t_{i-1}, t_i])) = \sup\{\beta(V(s)) : t_{i-1} \leq s \leq t_i\} = v(s_i) .
\]

On the other hand, by the mean value theorem we obtain
\[
Tu(t) = x_o + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} Fu(s)ds \in x_o + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{Conv}} Fu([t_i, t_{i+1}])
\]
for each $u \in V$. Therefore
\[
TV(t) \subset x_o + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{Conv}} F([V])([t_i, t_{i+1}]) .
\]
By (4) and the corresponding properties of \( \beta \) it follows that

\[
\beta(T(V)(t)) \leq \beta(x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i) \overline{\operatorname{Conv}} F([V])([t_i, t_{i+1}]))) \leq \\
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \beta(F(V)([t_i, t_{i+1}]))) \leq \\
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) r(\beta(V|_{[t_i, t_{i+1}]})) \leq \\
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) r(\beta(V(s_i))) \text{, for some } s_i \in [t_i, t_{i+1}] \\
= \sum_{i=0}^{m-1} (t_{i+1} - t_i) r(v(s_i)) .
\]

By letting \( m \to \infty \), we have

\[
\beta(T(V(t)) \leq \int_0^t r(v(s)) ds .
\] (5)

Since \( \overline{V} = \overline{\operatorname{Conv}}(\{x\} \cup T(V)) \) we have \( \beta(V(t)) \leq \beta(T(V(t))) \) and in view of (5), it follows that \( v(t) \leq \int_0^t r(v(s)) ds \) for \( t \in J \).

Hence applying now a theorem on differential inequalities (cf. [5]) we get \( v(t) = \beta(v(t)) = 0 \).

By Lemma 4, \( V \) is relatively weakly compact.

So, by Theorem 1 in [7], \( T \) has a fixed point in \( \overline{B} \) which is actually a pseudo-solution of (1).

As \( S = T(S) \), by repeating the above argument with \( V = S \) we can show that \( S \) is relatively compact in \((C(J, E), w))\).

Since \( T \) is weakly sequentially continuous on \( \overline{S(J)}^w \), \( S \) is weakly sequentially closed. By Eberlein-Smulian Theorem [3], \( S \) is weakly compact.

**Remark 6.** One can easily prove that the integral of a weakly continuous function is weakly differentiable with respect to the right endpoint of the integration interval and its derivative equals the integral at the same point (see [6], Lemma 2.3). In this case a pseudo-solution is, actually, a weak solution. Moreover, in some classes of spaces our pseudo-solutions are also strong \( C \)-solutions (in separable Banach spaces, for instance).

**References**


