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The Solvability Conditions of the Infinite Trigonometric Moment Problem with Gaps

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Abstract. The infinite Markov trigonometric moment problem with periodic gaps is considered. The precise analytical description of the solvability set of the problem is given. The introduced approach is based on investigation of the special subclass of the Carathéodory function class corresponding to given periodic law.

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Let p be a natural number and $M = \{m_0, \dots, m_\nu\}$, where

$$0 \leq m_0 < m_1 < m_2 < \cdots < m_\nu < p,$$

is the subset of the set $\{0, 1, \dots p-1\}$.

Definition 1. We will refer to the sequence $\overline{M} = \{m_k\}_{k=0}^{\infty}$, which is the p-periodic extension of M to the set $N \cup \{0\}$, i.e.

$$0 \leq m_0 < m_1 < \cdots < m_\nu < p \leq m_{\nu+1} = m_0 + p < \cdots$$
$$\cdots < m_{2\nu+1} = m_\nu + p < \cdots ,$$

as a p-periodic law generated by M.

If from $l \in M$ implies $p - l \in M$, we will say, that the p-periodic law is a symmetric one.

Let $\overline{M} = \{m_k\}_{k=0}^{\infty}$ be a p-symmetric periodic law. Consider the infinite Markov trigonometric moment problem of the form:

$$\int_0^\theta e^{im_k t} f(t)\, dt = s_{m_k}, \quad |f(t)| \leq 1, \qquad t \in (0, \theta),\ k = 0, 1, 2, \dots , \tag{1}$$

This is the preliminary version of the paper.

where $0 < \theta < \frac{2\pi}{p}$.

Our first goal is to give conditions for the complex sequence $\{s_{m_k}\}_{k=0}^{\infty} = \{a_{m_k} + ib_{m_k}\}_{k=0}^{\infty}$ (if $0 \in \overline{M}$ then $b_0 = 0$) to be a moment one. That means that there exists at least one measurable function f satisfying moment equalities (1).

As it is known, the classical trigonometric moment problem ($m_k = k$) is closely connected with Carathéodory coefficient problem [1] and based on the technique of the Carathéodory functions [2].

Remind that for the class C of Carathéodory functions one can write:

$$C := \{F\colon F \text{ is holomorphic, } \operatorname{Re} F(z) > 0 \text{ for } |z| < 1\}\,.$$

Further we need the following theorem describing properties of certain functions from this class:

Theorem 2. *Let* $T = \bigcup_{j=1}^{N} T_j$ *be a collection of nonintersecting intervals* $T_j = \left(\tau_j, \tau_j'\right) \subset [0, 2\pi]$. *Then the following statements are equivalent to each other:*

i) *A function* $F(z) \in C$ *is holomorphic for* $z = e^{i\tau}$ *and* $ImF(e^{i\tau}) = 0$, $\tau \in T$.

ii) *The following representation holds:*

$$F(z) = |F(0)| \exp\left\{ \frac{i}{4} \int\limits_{[0,2\pi]\setminus T} \frac{e^{it} + z}{e^{it} - z} \varphi\,(t)\, dt \right\},$$

$$-1 \le \varphi(t) \le 1,\;\; t \in [0, 2\pi]\setminus T.$$

iii) *Two functions*

$$F^{\pm}(z) = F(z) \cdot \left(\prod_{j=1}^{N} \frac{e^{i\tau_j'} - z}{e^{i\tau_j} - z} \cdot e^{-\frac{i}{2}\sum\limits_{j=1}^{N}\left(\tau_j' - \tau_j\right)} \right)^{\pm\frac{1}{2}}$$

belong to the class C *.*

Introduce the subclass of the Carathéodory function class associated with a p-periodic law.

Definition 3. For the subclass C($\overline{M}$) corresponding to the periodic law $\overline{M}$ we call the set of functions F(z), satisfying the following conditions:

i) $F \in C$.

ii) F is holomorphic and real on the arc $z = e^{i\tau}$, where $\tau \in \left(\frac{2\pi}{p}\,(p-\nu)\,, 2\pi\right)$.

iii) Power series for the function $\ln\left(\frac{F(z)}{|F(0)|}\right)$ is of the form:

$$\ln\left(\frac{F(z)}{|F(0)|}\right) = \sum_{k=0}^{\infty} \rho_k z^{m_k}, \qquad |z| < 1.$$

Further we give a multiplicative description of the class $C(\overline{M})$.

Introduce the polynomial $r^M(z)$ of the form:

$$r^M(z) = \prod_{k=1}^{q} \left(1 - e_p^{-\gamma_k} z\right) = \sum_{l=0}^{p-1} r_l z^l,$$

where e_p is a primitive root of unity, of order p,

$$\Gamma = \{\gamma_1, \dots, \gamma_q\} = \{0, 1, \dots, p-1\} \setminus M.$$

Note that $r_0 = 1$, $r_l = 0$, $l > q = p - \nu - 1$.

Besides, if $\overline{M}$ is a symmetric law then r_k are real, $k = 0, 1, \dots, p-1$.

Theorem 4. *A function $F(z) \in C(\overline{M})$, where $\overline{M}$ is a p-periodic symmetric law, iff*

$$F(z) = |F(0)| \exp\left\{ \frac{i}{4} \int_0^{\frac{2\pi}{p}} \sum_{l=0}^{q} r_l \frac{e^{i\left(t + \frac{2\pi}{p} l\right)} + z}{e^{i\left(t + \frac{2\pi}{p} l\right)} - z} \varphi(t)\, dt \right\},$$

where $-\mu \le \varphi(t) \le \mu$, $\mu^{-1} = \max\left\{|r_l|,\ l = \overline{0, q}\right\}$.

Let a sequence $\{s_{m_k}\}_{k=0}^{\infty}$ be a moment one for the problem (1). Complete then the definition of the function $f(t)$ by $f(t) = 0$, $t \in \left(0, \frac{2\pi}{p}\right)$. Next consider the function $\varphi(t)$, $t \in (0, 2\pi)$, of the form:

$$\varphi(t) = \mu r_l\, f\left(t - \frac{2\pi}{p} l\right), \quad t \in \left(\frac{2\pi}{p} l, \frac{2\pi}{p}(l+1)\right) = \Delta_l, \quad l = 0, 1, \dots p-1.$$

Note that

$$\begin{aligned} |\varphi(t)| &\le 1, \qquad t \in (0, 2\pi), \\ |\varphi(t)| &\le \mu, \qquad t \in \Delta_0; \end{aligned}$$

Next consider the complex function

$$F(z) = \exp\left\{ \frac{i}{4} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \varphi(t)\, dt \right\}. \tag{2}$$

Note that $|F(0)| = 1$. Hence due to Theorem 4 obtain that $F(z) \in C(\overline{M})$. Besides $\varphi(\tau) \equiv 0$, $\tau \in \left(\theta, \frac{2\pi}{p}\right)$, then the function $F(z)$ is holomorphic and real on the arc $z = e^{i\tau}$, $\tau \in \left(\theta, \frac{2\pi}{p}\right)$ (Theorem 2). Applying Theorem 2 once more and considering $T = \left(\theta, \frac{2\pi}{p}\right) \cup \left(\frac{2\pi}{p}(q+1), 2\pi\right)$, we obtain that $F^{\pm}(z) \in C$, where

$$F^{\pm}(z) = F(z) \cdot \left(\frac{e_p - z}{e^{i\theta} - z} \cdot \frac{1 - z}{e_p^{q+1} - z} \cdot exp\left\{ -\frac{i}{2} \left(\frac{2\pi}{p}(p - q) - \theta \right) \right\} \right)^{\pm\frac{1}{2}}. \tag{3}$$

Let

$$F^{\pm}(z) = \alpha^{\pm} + \sum_{j=1}^{\infty} \alpha_j^{\pm} z^j , \qquad |z| < 1. \tag{4}$$

Express now coefficients $\alpha^{\pm}, \alpha_j^{\pm}, j = 1, 2, \ldots$ of the expansion via elements of the moment sequence s_{m_k}, $k = 0, 1, \ldots$

Let

$$\ln F^{\pm}(z) = \sum_{j=0}^{\infty} s_j^{\pm} z^j, \tag{5}$$

then

$$s_j^{\pm} = \begin{cases} \frac{i\mu}{2}(2 - \delta_{0j}) r^M (e_p^j)\overline{s_j} \pm n_j, & j \in M, \\ \pm n_j, & j \notin M, \end{cases} \tag{6}$$

where

$$\begin{aligned} n_0 &= \frac{i}{4}\left(\frac{2\pi}{p}(p-q) - \theta\right), \\ n_j &= \frac{e^{-i\theta} + e^{-\frac{2\pi}{p}(q+1)i} - e^{-\frac{2\pi}{p}i} - 1}{j}, \end{aligned} \tag{7}$$

One can see from (4) and (6) that:

$$\alpha_j^{\pm} = \exp s_0^{\pm},$$

$$\mathcal{A}_j^{\pm} = \begin{vmatrix} \alpha_1^{\pm} & 2\alpha_2^{\pm} & \cdots & j\alpha_j^{\pm} \\ \alpha^{\pm} & \alpha_1^{\pm} & \cdots & \alpha_{j-1}^{\pm} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_1^{\pm} \end{vmatrix} = j\left(\alpha^{\pm}\right)^n s_j^{\pm}. \tag{8}$$

Thus, taking account of (6) we can regard (8) as recurrent expressions of the coefficients $\alpha^{\pm}, \alpha_j^{\pm}$, $j = 1, 2, \ldots$ via s_{m_k}, $k = 0, 1, 2 \ldots$

Now we are able to formulate the main result of the paper.

Theorem 5. *A sequence $\{s_{m_k}\}_{k=0}^{\infty}$ is a moment one for the Markov trigonometric moment problem (1) associated with the p-symmetric periodic law $\overline{M}$ iff*

$$A_n^{\pm} \geq 0, \; n = 0, 1, 2, \ldots$$

where

— the coefficients $\alpha^{\pm}, \alpha_j^{\pm}, j = 1, 2, \ldots$ are expressed via sequence $\left\{s_j^{\pm}\right\}_{j=0}^{\infty}$ by means of (8),

— $s_j^{\pm}$, $j = 0, 1, \ldots$ are of the form (6),

— n_j, $j = 0, 1, 2, \ldots$ are defined from (5),

— $A_n^{\pm}$, $n = 0, 1, \ldots$ are symmetric matrices such that $A_n^{\pm} = \left(\alpha_{k-j}^{\pm}\right)_{k,j=1}^{n}$,

where $\alpha_0^{\pm} = \alpha^{\pm} + \overline{\alpha}^{\pm} = 2\mathrm{Re}\, \alpha^{\pm}$, $\alpha_{-j}^{\pm} = \overline{\alpha_j^{\pm}}$, $j = 1, 2, \ldots$

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