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Elliptic Equations with Decreasing Nonlinearity I: Barrier method for Decreasing Solutions

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Abstract. In this note, we establish existence theorems for positive and classical solutions of the problem (Ea) below using a barrier method. Moreover we show that the existence of such solutions can be obtained from the sole existence of a supersolution or of a subsolution of the equation.

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1 Introduction

Let $f \in C^1([0,\infty)\times(0,\infty))$ be such that

f1) $\forall r \geq 0, \quad f(r,.)_+ := \max\{0, f(r,.)\} \in C^1((0,\infty))$ and non increasing;

f2) $\forall S, T > \theta > 0$, if $f(r,S), f(r,T) > 0$ then $\exists k_1(\theta), k_2(\theta) > 0$ such that $|f(r,T) - f(r,S)| \leq k_1(\theta) f_2(r, k_2(\theta))|T-S|$; $f_2(.,S) := |\partial f(.,S)/\partial S|$.

For $a > 1$, $p \in (1,2]$ and $D^p_a u := (r^a|u'|^{p-2}u')'$, consider in $\mathbb{R}_+$ the problem

$$Ea(u) := D^p_a u + r^a f(r,u)_+ = 0; \quad u(0) > 0;\ u'(0) = 0. \tag{Ea}$$

Definition 1. Let M be a positive number, finite or not. Let $I_M := [0, M)$ and $w, v \in C^1(\overline{I_M})$ be piecewise C^2 be non increasing.

1) v will be said to be a **supersolution (subsolution)** of the problem (Ea) in I_M if $Ea(v) \geq 0$ ($Ea(v) \leq 0$) almost everywhere in I_M;
2) w and v will be said to be **Ea-compatible** in I_M if
   i) $Ea(w) \leq 0 \leq Ea(v)$ a.e. in I_M,
   ii) $0 < w \leq v$ and $w' \leq v' \leq 0$ in I_M,
   iii) $\forall r \in I_M$, $f(r,.) > 0$ and decreasing in $[w(r), v(r)]$.

This is the final form of the paper.

For a non-increasing positive $\phi \in C^1(I_M)$ define

$$\Phi(r) = T\phi(r) := \phi(0) - \int_0^r dt\Big\{\int_0^t (s/t)^a f(s,\phi)ds\Big\}^{1/(p-1)}. \tag{T}$$

Definition 2. A non increasing (respectively decreasing) positive supersolution v (resp. subsolution w) of (Ea) in I_M will be said to be Ea-compatible if Tv and v (resp. w and Tw) are Ea-compatible in I_M.

In the sequel super- and subsolutions are supposed to be C^1 and piecewise C^2 in the corresponding domains. Also for ease writing, under the integral signs we will write $f(.,.)$ for $f(.,.)_+$. The main results are the following:

Theorem 3. *If there are w and v which are Ea-compatible in I_M, then (Ea) has a solution $u \in C^2(\overline{I_M})$ such that $w \le u \le v$ in I_M.*

Theorem 4. *Assume that there is a non increasing (resp. decreasing) positive supersolution v (resp. subsolution w) which is Ea-compatible in I_M.*
Then (Ea) has a decreasing solution $u \in C^2(\overline{I_M})$ such that $Tv \le u \le v$ (resp. $w \le u \le Tw$) in I_M.

Theorem 5. 1) *Assume that there are w and v which are Ea-compatible in $[0,\infty)$ such that*

$$\int_0^\infty \{1 + s^{p-1}\} f(s,w)ds < \infty. \tag{1}$$

Then (Ea) has a solution $u \in C^2([0,\infty))$ such that $w \le u \le v$ in $[0,\infty)$.
2) *Assume that there is a non increasing (resp. decreasing) positive supersolution v (resp. subsolution w) Ea-compatible in $\mathbb{R}_+$.*
Then (Ea) has a positive decreasing solution $u \in C^2([0,\infty))$ such that it holds $Tv \le u \le v$ (resp. $w \le u \le Tw$) in $[0,\infty)$.

Theorem 6. 1) *Assume that there are w and v which are Ea-compatible in $[0,\infty)$ with*

$$\int_0^\infty \{sf(s,w)\}^{1/(p-1)} < \infty. \tag{2}$$

Then (Ea) has a solution $u \in C^2([0,\infty))$ such that $w \le u \le v$.
2) *Assume that there is a non increasing positive supersolution v of (Ea) in $[0,\infty)$ such that*

i) *$V(r) = Iv(r) := \int_r^\infty dt\{\int_0^t (s/t)^a f(s,v)ds\}^{1/(p-1)}$ satisfies (2);*
ii) *V and v are Ea-compatible in $[0,\infty)$.*

Then (Ea) has such a solution u with $V \le u \le v$.
Similarily if there is a decreasing positive subsolution w such that w and Iw are Ea-compatible in $[0,\infty)$ and which satisfies (2), then (Ea) has such a solution u with $w \le u \le W := Iw$.

2 Proof of the theorems

2.1 Preliminaries

Let $C_f(M) := \{\phi \in C(\overline{I_M}) \mid f(r,\phi) > 0 \ \forall r \in I_M\}$ and $b := 1/(p-1)$. For some $A = \phi(0)$, define on $C_f(M)$ the operator T by

$$\Phi(r) := T\phi(r) := A - \int_0^r dt \left\{ \int_0^t (s/t)^a f(s,\phi) ds \right\}^b . \tag{3}$$

Then $D_a^p \Phi + r^a f(r,\phi) = 0 \quad \text{in} \quad \mathrm{I_M}, \quad \Phi(0) = \mathrm{A}, \quad \Phi'(0) = 0$ and $\Phi' \leq 0$. From [5], as $b \geq 1$, $\forall t \leq M$, with $s_* := \max\{1, s\}$,

$$|\Phi(t)| \leq \frac{p-1}{a+1-p} \left\{ \int_0^t s_*^{p-1} f(s,\phi) ds \right\}^b ; \tag{4}$$

$$|\Phi'(t)| \leq \frac{1}{t_*} \left\{ \int_0^t s_*^{p-1} f(s,\phi) ds \right\}^b . \tag{5}$$

As $\Phi''(t) = -b\{\int_0^t (s/t)^a f(s,\phi) ds\}^{b-1} \{f(t,\phi) - \frac{a}{t} \int_0^t (s/t)^a f(s,\phi) ds\}$,

$$|\Phi''(t)| \leq b \left\{ \int_0^t (s/t)^a f(s,\phi) ds \right\}^{b-1} \left\{ f(t,\phi) + \frac{a}{t} \int_0^t f(s,\phi) ds \right\}. \tag{6}$$

Thus $TC_f(M) \subset C^2(\overline{I_M})$ and for $\phi \in C_f(M)$,

$$\begin{aligned} |T\phi|_{C^2([0,M]} \leq \quad & C_M^2(\phi) := A + \frac{a}{a+1-p} \left\{ \int_0^M s_*^{p-1} f(s,\phi) ds \right\}^b + \\ & + b(a+1) |f(.,\phi)|_{C(I_M)} \left\{ \int_0^M f(s,\phi) ds \right\}^{b-1} . \end{aligned} \tag{7}$$

Lemma 7. *Let w, v be those in Theorem 3 and define*

$$E_M(w,v) := \{\phi \in C^1(I_M) \mid w \leq \phi \leq v; \ w' \leq \phi' \leq v' \ inI_M\}.$$

Then with $A \in [w(0), v(0)]$, $\quad TE_M(w,v) \subset E_M(w,v) \cap C^2(\overline{I_M})$.

Proof. Let $V := Tv$ and $W := Tw$; then in I_M

$$w \leq W \leq V \leq v \qquad \text{and} \qquad w' \leq W' \leq V' \leq v'.$$

In fact, as $V', v' \leq 0$, $D_a^p V - D_a^p v = (r^a\{|v'|^{p-1} - |V'|^{p-1}\})' \leq 0$ whence $|v'|^{p-1} \leq |V'|^{p-1}$ or $V' \leq v' \leq 0$. Because $V(0) \leq v(0)$ we then have $V \leq v$ in I_M. Similarily we have $w' \leq W'$ and $w \leq W$ in I_M. Also in the same way, $w \leq v$ and $W(0) = V(0)$ imply that $W' \leq V'$ and $W \leq V$. If $\phi \in E_M(w,v)$ then $f(r,v) \leq f(r,\phi) \leq f(r,w)$ in I_M, hence $\Phi := T\phi$ satisfies

$$W \leq \Phi \leq V \qquad \text{and} \qquad W' \leq \Phi' \leq V'.$$

Corollary 8. *Let v (w) be a non increasing (decreasing) positive supersolution (subsolution) which is Ea-compatible in I_M.*
Then $TE_M(v) \subset E_M(v) \cap C^2(\overline{I_M})$, where $E_M(v) \equiv E_M(Tv, v)$ ($TE_M(w) \subset E_M(w) \cap C^2(\overline{I_M})$, where $E_M(w) \equiv E_M(w, TW)$).

Proof. In the light of Lemma 7, it is enough to notice that $V := Tv$ ($W := Tw$) is a subsolution (supersolution) of (Ea) in I_M.

Lemma 9. *Let w and v be as in Theorem 3. Then, $T : E_M(w, v) \longrightarrow C^1(\overline{I_M})$ is continuous and $TE_M(w, v)$ is equicontinuous in $C^1(\overline{I_M})$.*

Proof. The continuity follows from the fact that for $\phi, \psi \in E_M(w, v)$ and $|\ |_r$ denoting the norm in $C([0, r])$,

$$|(|(T\phi)'|^{p-1} - |(T\psi)'|^{p-1})(t)| \leq k_1(\theta)|\phi - \psi|_r \left\{ \int_0^r (s/r)^a f_2(s, k_2(\theta)) \right\},$$

where $\phi, \psi > \theta > 0$ in I_M is assumed (see f2)) and a similar bound for $|T\phi - T\psi|$ is obtained easily. The equicontinuity in C^1 follows from (7).

2.2 Proof of Theorems 3 and 4

Lemma 7 and Lemma 9 imply that T has a fixed point in $E_M(w, v)$ by the Schauder-Tychonoff's fixed point theorem [2]; (6)–(7) imply that the fixed point is in $C^2(\overline{I_M})$. In the same way Corollary 8 and Lemma 9 imply that T has such a fixed point in $E_M(v)$ ($E_M(w)$).

2.3 Proof of Theorem 5

We prove 1) only as 2) and 3) would be simple readaptations.
If (1) holds, then $V := Tv$ and $W := Tw$ are in $E(w, v) \cap C^2([0, \infty))$. With (1), (4)–(7) imply that $\forall \phi \in E(w, v) := E_\infty(w, v)$,

$$|T\phi|_{C^2(I_M)} \leq C^2_\infty(w) \qquad \forall M > 0. \tag{8}$$

Let $(M_k)_{k\in\mathbb{N}}$ be an increasing sequence such that $M_k \nearrow \infty$ and ($u_k := u_{M_k}$) the corresponding solutions in $I_k := I_{M_k}$. u_k is extended by $\overline{u_k} := Tu_k \in C^2(\mathbb{R}_+)$, say, which satisfies (8) and $Ea(\overline{u_k}) = 0$ in I_k, $\overline{u_k}(0) = A$. By means of the Schauder-Tychonoff's fixed point theorem, such a required solution is an inductive limit of the ($\overline{u_k}$) ([3]).

2.4 Proof of Theorem 6

Define this time the inverse operator of (Ea) in I_M, $K := K_M$ on $C_f(M)$ by

$$\Phi(r) = K\phi(r) := \int_r^M dt \left\{ \int_0^t (s/t)^a f(s, \phi) ds \right\}^b.$$

From Jensen's inequality $\{(1/t)\int_o^t s^a f(s,\phi)ds\}^b \leq (1/t)\int_0^t \{s^a f(s,\phi)\}^b ds$ and simple integrations by parts, as in (4)–(7), $\forall t \in I_M$,

$$b(a-1)\,\Phi(t) \leq \int_0^M s^b f(s,\phi)^b ds := I_M^b(\phi); \qquad |\Phi'(t)| \leq (1/t) I_t^b(\phi)$$

and (6) holds for this case.
If necessary, we replace f by $f_1 := \lambda f$ such that

$$[(p-1)/(a-1)] \int_0^\infty \{s f_1(s,w)\}^{1/(p-1)} ds < v(0) \qquad \text{in (2);}$$

the required solution will be $u(r) := u_1(\mu r)$ for some suitable $\mu = \mu(\lambda)$, u_1 being obtained with f_1. So, without major difficulties the proof of this Theorem follows the same steps as that of Theorem 5.

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