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Linear differential Lappo-Danilevskii systems

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Abstract. The class of linear differential systems with coefficient matrices which commutative with their integrals is considered. The results on asymptotic equivalence of these systems and their distribution among linear systems are given.

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Consider the linear system

$$\frac{dx}{dt} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in I = [t_0, +\infty[, \tag{1}$$

where $A(t)$ is an $n \times n$ matrix of real-valued continuous and bounded functions of real variable t on the non-negative half-line I. Usually [1, p. 117], (1) is called Lappo-Danilevskii's system if the matrix A is commutative with its integral, i.e.

$$A(t)\int_s^t A(u)du = \int_s^t A(u)du A(t), \tag{2}$$

for some s, $t \in I$.

We define three types of the Lappo-Danilevskii systems.

This is author's version of the invited lecture.

Definition 1. We say that

i) $A(t)$ is a right Lappo-Danilevskii matrix with the initial point s ($A \in LD_r(s)$) if there exists an s, $s \in I$, such that (2) is fulfilled for all $t \in I_r(s) = [s, +\infty[$;

ii) $A(t)$ is a left Lappo-Danilevskii matrix with the initial point s ($A \in LD_l(s)$) if there exists an $s \in I$, $s > t_0$, such that (2) is fulfilled for all $t \in I_l(s) = [t_0, s]$;

iii) $A(t)$ is a bilateral Lappo-Danilevskii matrix with the initial point s ($A \in LD_b(s)$) if there exists an s, $s \geq t_0$, such that (2) is fulfilled for all $t \in I$.

The corresponding systems (1) are called right, left or bilateral Lappo-Danilevskii systems. Note that a special case of the bilateral Lappo-Danilevskii system is system (1) with the functional commutative matrix A, where for all $s, t \in I$

$$A(t)A(s) - A(s)A(t) = 0. \tag{3}$$

It is well known that if A is a right, left or bilateral Lappo-Danilevskii matrix, then a fundamental solution matrix $X_s(t)$ of (1) ($X_s(s) = E$, E is the identity matrix) can be represented as

$$X_s(t) = \exp \int_s^t A(u)du \tag{4}$$

for $t \in I_r(s)$, $t \in I_l(s)$, $t \in I$ respectively. This simple representation (4) of the fundamental solution matrix does explain the fact that the class of Lappo-Danilevskii systems is one of the main and interesting class of linear systems. For example, in some cases it is possible to calculate asymptotic characteristics, in particular, Lyapunov exponents of the solutions of (1) directly using coefficients of (1) (see for instance [2]). In this connection we consider a problem of reducibility of an arbitrary linear system with bounded coefficients to the Lappo-Danilevskii system and to the system with functional commutative matrix of coefficients.

It is well known [3, p. 274] that any linear system is an almost reducible to some diagonal system. It is trivial fact that any diagonal matrix is a functional commutative matrix. But quite different is the case of linear systems under Lyapunov's transformations.

A linear transformation

$$x = L(t)y \tag{5}$$

is a Lyapunov transformation if $L(t)$ is a Lyapunov matrix, i.e.

$$\max\{\sup_{t\geq t_0} \|L(t)\|, \sup_{t\geq t_0} \|L^{-1}(t)\|, \sup_{t\geq t_0} \|\frac{d}{dt}L(t)\|\} < +\infty. \tag{6}$$

It is easy to see that if (5) reduces (1) to the system

$$\frac{dy}{dt} = B(t)y, \quad y \in \mathbb{R}^n, \quad t \in I, \tag{7}$$

then

$$B(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\frac{d}{dt}L(t). \tag{8}$$

We follow Yu. Bogdanov [4] and say that two linear systems are asymptotically equivalent if there exists a Lyapunov transformation reducing one of them to the other. Note that the Lyapunov transformations do not change asymptotic properties of the solutions, in particular, their stability.

Theorem 1. *The linear system (7) is asymptotically equivalent to the system (1) with functional commutative matrix of coefficients if and only if the Cauchy matrix $K_B(t,s)$ of (7) can be presented in the form*

$$K_B(t,s) = L(t)\exp\int_s^t A(u)du L^{-1}(s) \quad \forall t,s \geq t_0, \tag{9}$$

where $L(t)$ is Lyapunov's matrix.

Proof. 1. Let (7) be asymptotically equivalent to (1) with the functional commutative matrix A satisfying (3). Then there exist $s_0 \geq t_0$ and non-singular constant matrix C such that

$$Y_{s_0}(t)CX_{s_0}^{-1}(t) = L(t), \tag{10}$$

where $L(t)$ is Lyapunov's matrix, X_{s_0} and Y_{s_0} are fundamental matrices of the solutions of (1) and (7) respectively ($X_{s_0}(s_0) = Y_{s_0}(s_0) = E$). Since A is a functional commutative matrix, we have

$$K_A(t,s) = \exp\int_s^t A(u)du, \tag{11}$$

where $K_A(t,s)$ is the Cauchy matrix of (1). From (10) it follows that

$$K_B(t,s) = Y_{s_0}(t)Y_{s_0}^{-1}(s) = L(t)X_{s_0}(t)C^{-1}CX_{s_0}^{-1}(s)L^{-1}(s) = L(t)K_A(t,s)L^{-1}(s).$$

Using (11), we obtain the required relation (9).

2. Let transformation

$$y = L(t)x \tag{12}$$

with the Lyapunov matrix $L(t)$ satisfying (6) reduce (7) to some linear system $Dx = P(t)x$. Then P satisfies (see (8)) the equality

$$P(t) = L^{-1}(t)B(t)L(t) - L^{-1}(t)\frac{d}{dt}L(t).$$

Since $L(t) = K_B(t,s)L(s)\exp\Big(-\int_s^t A(u)du\Big)$, we have

$$\frac{d}{dt}L(t) = B(t)K_B(t,s)L(s)\exp\Big(-\int_s^t A(u)du\Big) -$$

$$-K_B(t,s)L(s)\exp\Big(-\int_s^t A(u)du\Big)D\Big(\exp\int_s^t A(u)du\Big)\exp\Big(-\int_s^t A(u)du\Big)=$$

$$=B(t)L(t)-L(t)\frac{d}{dt}\Big(\exp\int_s^t A(u)du\Big)\exp\Big(-\int_s^t A(u)du\Big).$$

Therefore, $P(t)=\frac{d}{dt}\Big(\exp\int_s^t A(u)du\Big)\exp\Big(-\int_s^t A(u)du\Big)$, hence

$$P(t)\exp\int_s^t A(u)du=\frac{d}{dt}\Big(\exp\int_s^t A(u)du\Big)\quad \forall t,s\geq t_0.$$

Thus,

$$P(t)\Big(E+\sum_{m=1}^{\infty}\frac{1}{m!}\Big(\int_s^t A(u)du\Big)^m\Big)=\frac{d}{dt}\Big(E+\sum_{m=1}^{\infty}\frac{1}{m!}\Big(\int_s^t A(u)du\Big)^m\Big)=$$

$$=A(t)+\sum_{m=2}^{\infty}\frac{1}{m!}\sum_{k=0}^{m-1}\Big(\int_s^t A(u)du\Big)^k A(t)\Big(\int_s^t A(u)du\Big)^{m-1-k}\quad \forall t,s\geq t_0.$$

Substituting t for s, we get $P(t)=A(t)$ for all $t\geq t_0$. Therefore, (12) reduces (7) to (1). It suffices to show that A is a functional commutative matrix.

Consider transformation (12); if Y is any fundamental matrix of the solutions of (7), then $X(t)=L^{-1}(t)Y(t)$ is the fundamental matrix of (1). Therefore, from (9) it follows that

$$K_A(t,s)=X(t)X^{-1}(s)=L^{-1}(t)Y(t)Y^{-1}(s)L(s)=$$

$$=L^{-1}(t)K_B(t,s)L(s)=\exp\int_s^t A(u)du\qquad \forall t,s\geq t_0.$$

From [5] it follows that A is a functional commutative matrix. The theorem is proved.

The similar result is valid for the right and bilateral Lappo-Danilevskii systems.

Theorem 2. *The linear system* (7) *is asymptotically equivalent to the right* (bilateral) *Lappo-Danilevskii system* (1) *if and only if there exists a fundamental matrix* $Y(t)$ *of* (7) *which can be presented in the form*

$$Y(t)=L(t)\exp\int_s^t A(u)du\quad \forall t\geq s\geq t_0\ (\forall t\geq t_0),$$

where $L(t)$ *is Lyapunov's matrix and* $A\in LD_r(s)$ $(A\in LD_b(s))$.

There is no problem to reduce linear systems to left Lappo-Danilevskii systems, because it is easy to prove that any linear system is asymptotically equivalent to some left Lappo-Danilevskii system. On the other hand, the following result is valid [6,7].

Theorem 3. *There exists a linear system which is asymptotically equivalent neither to any system with functional commutative matrix of coefficients nor to any right (bilateral) Lappo-Danilevskii system.*

To prove this fact it is sufficient to consider the linear system with the following matrix of coefficients (E_{n-2} is the $(n-2)\times(n-2)$ identity matrix)

$$\begin{pmatrix} 0 & 1 & 0 \,...\, 0 \\ 0 & (t-t_0+1)^{-1} & 0 \,...\, 0 \\ 0 & 0 & \\ \cdots & \cdots & E_{n-2} \\ 0 & 0 & \end{pmatrix}, \quad t\in[t_0,+\infty[, \tag{13}$$

and to use the specific structure and the distribution of zeros of the integrals of the Lappo-Danilevskii matrices.

However, system (13) is a regular system (in the Lyapunov sense) and it can be reduced (Basov-Grobman-Bogdanov's criterion [8, p. 77] to the system with functional commutative coefficients by generalized Lyapunov transformation (5) with the matrix L such that $\overline{\lim\limits_{t\to+\infty}}\, t^{-1}\ln\|L(t)\| = \overline{\lim\limits_{t\to+\infty}}\, t^{-1}\ln\|L^{-1}(t)\| = 0.$

But even if we expand the set of our transformations up to the set of generalized Lyapunov transformations there is a statement which is similar to Theorem 3 [9,10].

Theorem 4. *There exists a two-dimensional linear system which is generalized asymptotically equivalent neither to any system with functional commutative matrix of coefficients nor to any right (bilateral) Lappo-Danilevskii system.*

We think that this fact holds for the linear systems of the arbitrary dimension n, but now we have the proof only for $n=2$.

Note that condition (2) is sufficiently strong and small perturbations of the elements of A can output the matrix from the class of the Lappo-Danilevskii matrices. So we consider some problems on the behavior of the Lappo-Danilevskii matrices in the set of all matrices.

Let the distance between matrices A and B be defined by the following formula $\rho(A,B)=\sup_{t\geq t_0}\|A(t)-B(t)\|$, where $\|\cdot\|$ be an arbitrary matrix norm.

The following results are fulfilled [11].

Theorem 5. *Let $A_i\in LD_\alpha(s_i)$, $i\in\mathbb{N}$, $\alpha\in\{b,r\}$, and $\rho(A,A_i)\to 0$ as $i\to+\infty$. If there exists M such that $s_i\leq M<+\infty$ for all $i\in\mathbb{N}$, then A is a bilateral (right) Lappo-Danilevskii matrix.*

Theorem 6. *Let $A_i \in LD_l(s_i)$, $i \in \mathbb{N}$, and $\rho(A, A_i) \to 0$ as $i \to +\infty$. If there exist m, M such that $t_0 < m \le s_i \le M < +\infty$ for all $i \in \mathbb{N}$, then A is a left Lappo-Danilevskii matrix.*

However, if the sequences (s_i) for the sequences of right and left Lappo-Danilevskii matrices are not bounded, then the previous results are not valid, namely, the following facts hold.

Theorem 7. *There exists a sequence A_i, $A_i \in LD_r(s_i)$, ($A_i \in LD_l(s_i)$,) $i \in \mathbb{N}$, $\rho(A, A_i) \to 0$ and $s_i \to +\infty$ ($s_i \to t_0 + 0$) as $i \to +\infty$, such that $A \notin LD_r$ ($A \notin LD_l$.)*

To prove this statement it is sufficient to construct the following sequences of A_k ($k \in \mathbb{N}$, $t_0 = 0$):

$$A_k(t) = \begin{pmatrix} B_k(t) & O_1 \\ O_2 & C(t) \end{pmatrix}, \quad B_k(t) = \begin{pmatrix} g(t) & f_k(t) \\ e^{-t} & g(t) \end{pmatrix}, \quad t \in [0, +\infty[,$$

where O_1, O_2 are the $2 \times (n-2)$, $(n-2) \times 2$, zero-matrices respectively, $C(t)$ is an $(n-2) \times (n-2)$ functional commutative matrix, g is a continuous bounded function on $[0, +\infty[$. If

$$f_k = \begin{cases} (1 - e^{-t})e^{-t}, \ 0 \le t \le k, \\ (1 - e^{-k})e^{-t}, \quad t > k, \end{cases}$$

then $A_k \in LD_r(k)$, but the limit matrix A does not belong to LD_r; if

$$f_k = \begin{cases} e^{-\frac{1}{k} - t}, \ 0 \le t \le \dfrac{1}{k}, \\ e^{-2t}, \qquad t > \dfrac{1}{k}, \end{cases}$$

then $A_k \in LD_l(\frac{1}{k})$, but the limit matrix A does not belong to LD_l.

The following result establishes the closure of the set of two dimensional bilateral Lappo-Danilevskii matrices in the set of all matrices.

Theorem 8. *Let $A_i \in LD_b(s_i)$, $i \in \mathbb{N}$. If $\rho(A, A_i) \to 0$ as $i \to +\infty$, then A is a bilateral Lappo-Danilevskii matrix.*

To complete our review of the Lappo-Danilevskii systems we say some words about connection of the properties (2) and (4).

It is well known that condition (2) is sufficient for the representation (4). J. F. P. Martin proved (see [12]) that if the differences of the eigenvalues of the integral of A were not zero roots of the equation

$$e^z - z - 1 = 0, \tag{14}$$

then (4) implied (2). From the results of J.F.P. Martin [12] and V.N. Laptinskii [13] it follows that if coefficients of (1) are analytic functions on I, then (4) also implies (2). However, it was open question on the existence of a linear system with infinitely differentiable non-analytic coefficients such that this system was not a Lappo-Danilevskii system but its fundamental solution matrix had the form (4). We proved that such system exists [14].

To verify this fact it is sufficient to consider system (1) with the matrix

$$A(t) = \begin{pmatrix} -\mu a(t) & 0 & -\nu a(t) \\ b(t) & 0 & 0 \\ \nu a(t) & 0 & -\mu a(t) \end{pmatrix}, \quad t \in [0, +\infty[,$$

where $\mu \pm i\nu$ are roots of the equation (14), a and b are infinitely differentiable non-analytic functions such that

$$\int\limits_0^t a(u)du > 0 \;\; \forall\, t \in]0, s_0], \quad \int\limits_0^{s_0} a(u)du = 1, \qquad a(t) = 0 \;\; \forall\, t \geq s_0 > 0, \tag{15}$$

$$b(t) = \begin{cases} 0, & t \in [0, s_0[, \\ b_k(t) \not\equiv 0, & t \in [s_{2k}, s_{2k+1}[, \\ 0 & t \in [s_{2k+1}, s_{2k+2}[,\ k = 0, 1, ..., \end{cases} \tag{16}$$

((s_k) is an arbitrary sequence of positive numbers such that $s_{k+1} > s_k$ and $s_k \to +\infty$ as $k \to +\infty$). In this case the fundamental solution matrix $X_0(t)$ of (1) may be represented as (4) with $s = 0$ but $A(t)$ is not a Lappo-Danilevskii matrix with initial point $s = 0$.

Note that for two dimensional real-valued matrix A condition (2) is necessary and sufficient for representation (4), it follows from the distributions of the roots of (14) and the eigenvalues of the integral of A. But for two dimensional complex-valued matrix A condition (2) is not necessary for (4). For example, if γ is a root of (14) and the functions a, b satisfy (15) and (16), then the matrix

$$A(t) = \begin{pmatrix} -\frac{\gamma a(t)}{2} & 0 \\ b(t) & \frac{\gamma a(t)}{2} \end{pmatrix}, \quad t \in [0, +\infty[,$$

is not Lappo-Danilevskii's matrix with initial point $s = 0$, however the fundamental solution matrix $X_0(t)$ of (1) may be represented as (4) with $s = 0$.

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