Veronika Chrastinová

# A general controllability theorem

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 103--108.

Persistent URL: http://dml.cz/dmlcz/700340

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 103–108

# A general controllability theorem

Veronika Chrastinová ⋆

Department of Mathematics and Descriptive Geometry,
Faculty of Civil Engineering, Technical University,
Žižkova 17, 662 37 Brno, Czech Republic,
Email: Chrastinova.V@fce.vutbr.cz

**Abstract.** The report concerns a fundamental result from the theory of quite general systems $\Omega$ of smooth partial differential equations. The existence of a unique "composition series" of the kind $\Omega^0 \subset \Omega^1 \subset \ldots \subset \Omega$ consisting of "factorsystems" of $\Omega$ is explained. Here $\Omega^k$ is the maximal system of differential equations"induced" by $\Omega$ such that the formal solution of $\Omega^k$ depends on the choice of arbitrary functions of $k$ variables (on constants if $k = 0$). This is a well-known result only in the particular case of underdetermined systems of ordinary differential equations. Then $\Omega^1 = \Omega$ and $\Omega^0$ involves all first integrals, hence $\Omega^0$ is trivial if and only if $\Omega$ is a controllable system. In full generality, we may speak of a "multidimensional controllability" composition series of $\Omega$.

**MSC 2000.** 35A30, 58A17, 58E99

**Keywords.** Underdetermined, determined and overdetermined differential equations, controllability, composition series.

## 1 Introduction

The article concerns certain aspects of the formal theory of general compatible systems $\Omega$ of smooth partial differential equations

$$\Omega : \, f_i\left(x_1, \ldots, x_n, u^1, \ldots, u^m, \ldots, \frac{\partial^{i_1+\ldots+i_n} u^k}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \ldots\right) = 0 \quad (i = 1, \ldots, I)$$

⋆ This research has been conducted at the Department of Mathematics and Descriptive Geometry as part of the research project CEZ : MSM J22 : 261100009.

*This is the preliminary version of the paper.*

which can be closely related to the classical calculus of variations. Let us therefore briefly recall the general *Lagrange problem* of the calculus of variations: a solution $u^1, \ldots, u^m$ of $\Omega$ in a domain $\mathcal{D}$ and satisfying certain boundary conditions is to be determined such that a given integral

$$\int_{\mathcal{D}} f\left(x_1, \ldots, x_n, u^1, \ldots, u^m, \ldots, \frac{\partial^{i_1+\ldots+i_n} u^k}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \ldots\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

attains the extremal value. It is well-known that the sought extremal solution $u^1, \ldots, u^m$ obey the familiar necessary conditions, the Euler-Lagrange system

$$\frac{\partial F}{\partial u^k} - \sum \frac{\mathrm{d}}{\mathrm{d}x_i} \cdot \frac{\partial F}{\partial(\partial u^k/\partial x_i)} + \sum \frac{\mathrm{d}^2}{\mathrm{d}x_i \mathrm{d}x_j} \cdot \frac{\partial F}{\partial(\partial^2 u^k/\partial x_i \partial x_j)} - \ldots = 0\,,$$

$$F = f + \sum \lambda^i f_i\,, \quad \lambda^i = \lambda^i(x_1, \ldots, x_n)\,, \quad k = 1, \ldots, m\,,$$

where $\lambda^i$ are certain unknown coefficients, the *Lagrange multipliers.* In the favourable case when the system $\Omega$ is empty, the functions $f_i$ and hence the multipliers are absent and then the theory very simplifies – this case can be found in current textbooks. In general, the presence uf uncertain multipliers $\lambda^i$ causes many difficulties, especially for the so-called degenerate variational problems.

The question may be asked whether the multipliers $\lambda^i$ can be explicitly expressed in terms of the functions $u^k$ and their derivatives. In the one-dimensional case for $n = 1$, $x = x_1$ of ordinary differential equations $\Omega$ the answer to the question is known: this is possible if and only if the system $\Omega$ does not admit any nontrivial first integrals, i. e., functions that are constant on every solution of $\Omega$. In the terminology of the optimal regulation theory, $\Omega$ is called a *controllable system* in this case.

Our idea is as follows: in order to investigate the nature of Lagrange multipliers in the multidimensional case with the number of independent variables $n > 1$, it is desirable to study the generalized controllability concept for a general system $\Omega$.

## 2 Classification of equations

Eventually passing to the proper topic, let us recall some concepts concerning the formal theory of general systems $\Omega$. Such a system need not have any actual solutions at all but it always has the *formal solutions* represented by formal power series, i. e., the values of derivatives of the unknown functions calculated at a fixed point. We shall denote $\Omega = \Omega^\nu$ if this formal solution depends on the choice of certain number $\mu = \mu(\Omega) \geq 1$ of arbitrary functions of $\nu = \nu(\Omega)$ independent variables (on constants in $\nu = 0$). Recall that the formal solutions may be identified with the actual solutions in favourable cases, e. g., for analytical systems $\Omega$. For better clarity, let us mention some examples.

First assume the ordinary differential equations, hence $n = 1$, $x = x_1$. They can be always represented be a *first order* system and then two subcases are distinguished:

1. Determined system $\Omega = \Omega^0$:

$$\frac{\mathrm{d}u^i}{\mathrm{d}x} = f_i(x, u^1, \dots, u^m)\,; \quad i = 1, \dots, m\,;$$

   - number of equations equals the number of unknown functions,
   - solution depends on constants $u^1(x_0) = c_1, \dots, u^m(x_0) = c_m$,
   - investigated in common theory of ordinary differential equations.

2. Underdetermined system $\Omega = \Omega^1$ with $0 \le a \le m$:

$$\frac{\mathrm{d}u^i}{\mathrm{d}x} = f_i\left(x, u^1, \dots, u^a, u^{a+1}, \dots, u^m, \frac{\mathrm{d}u^{a+1}}{\mathrm{d}x}, \dots \frac{\mathrm{d}u^m}{\mathrm{d}x}\right)\,; \quad i = 1, \dots, a\,;$$

   - number of equations is less than the number of unknown functions,
   - $u^{a+1}(x), \dots, u^m(x)$ can be arbitrarily chosen,
   - needful in the calculus of variations.

Second, let us state only very particular examples for the case of two independent variables $n = 2$, denoting $x = x_1$, $y = x_2$. Then three subcases should be distinguished: overdetermined, determined or underdetermined system $\Omega$. It is defined by the property that the solution depends either on mere constants, or on the choice of certain functions of one independent variable, or on certain arbitrary functions of two variables.

1. Overdetermined system $\Omega = \Omega^0$:

$$\frac{\partial u^j}{\partial x} = f_j(x, y, u^1, \dots, u^m)\,; \quad \frac{\partial u^j}{\partial y} = g_j(x, y, u^1, \dots, u^m)\,; \quad j = 1, \dots, m\,;$$

   - number of equations is greater than the number of unknown functions,
   - values $u^1(x_0, y_0) = c_1, \dots, u^m(x_0, y_0) = c_m$ uniquely determine the solution,
   - occuring e. g. in the theory of Lie groups and differential geometry.

2. Determined system $\Omega = \Omega^1$ (Cauchy - Kowalewska system):

$$\frac{\partial u^j}{\partial x} = f_j(x, y, u^1, \dots, u^m, \frac{\partial u^1}{\partial y}, \dots, \frac{\partial u^m}{\partial y})\,, \quad j = 1, \dots, m\,;$$

   - number of equations equals the number of unknown functions,
   - initial values $u^1(x_0, y), \dots, u^m(x_0, y)$ determine the solution,
   - investigated in the common theory of partial differential equations (especially in matematical physics).

3. Underdetermined system $\Omega = \Omega^2$:

$$\frac{\partial u^j}{\partial x} = f_j(x, y, u^1, \dots, u^m, \frac{\partial u^{a+1}}{\partial x}, \dots, \frac{\partial u^m}{\partial x}, \frac{\partial u^1}{\partial y}, \dots, \frac{\partial u^m}{\partial y})\,;$$
$$j = 1, \dots, a \quad (0 \le a < m)$$

- number of equations is less than the number of unknown functions,
- $u^{a+1}(x,y),\dots,u^m(x,y)$ can be arbitrarily chosen,
- needful in the calculus of variations.

We shall not state other examples for general $n>2$. As for the terminology, let us only recall that always $\nu\le n$ and it is useful to classify three subcases:

$\Omega=\Omega^n$ *underdetermined systems* (calculus of variations),
$\Omega=\Omega^{n-1}$ *determined systems* (mathematical physics),
$\Omega=\Omega^a,\ a<n-1$ *overdetermined systems* (Lie groups, differential geometry).

Unfortunately, this terminology is not commonly accepted.

## 3 Controllability theorem

After these somewhat lengthy preparations, let us eventually pass to the main result:

*Controllability theorem. Let $\Omega=\Omega^\nu$ be a system such that the formal solution depends on a certain number of arbitrary functions of $\nu$ variables, on constants if $\nu=0$. Then for every $a$, $0\le a<\nu$, there exists a unique maximal subsystem $\Omega^a\subset\Omega$ which is induced in a certain sense by $\Omega$ such that its solutions depend on the choice of arbitrary functions of $a$ independent variables. So we have a unique composition series*

$$\Omega^0\subset\Omega^1\subset\dots\Omega^a\subset\dots\subset\Omega^\nu\quad(a<\nu)\,.$$

The initial term $\Omega^0$ involves all first integrals $F^i$ of $\Omega^\nu$, i.e., the system $\Omega^0$ can be represented by certain Pfaffian equations $\mathrm{d}F^i=0$ where $F^i$ are appropriate functions. This exactly corresponds to the *controllability* concept for the particular case $n=1$, $\nu=1$ of ordinary differential equations, but in general we have a far going generalization of this well-known achievement. If all $\Omega^0,\dots,\Omega^{\nu-1}$ are trivial, we may speak of the generalized controllable system $\Omega$.

We cannot discuss here the proof in more details because rather unusual tools must be employed and instead refer to the literature below. Let us note only that the terms $\Omega^a$ can be determined by a pure algebraic computation. No deep existence theorems concerning the solutions of partial differential equations are necessary in the proofs and the same algorithm can be applied if one tries to calculate the Lagrange multipliers of a variational problem.

## 4 Two examples

Let us conclude with two quite simple examples for the case ot two independent variables ($n=2$). We use slightly simplified notation here.

1. We start with the determined system of equations

$$\Omega = \Omega^1 \,:\, F\left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = G\left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = 0\,,$$

$$\operatorname{rank}\begin{pmatrix} \partial F/\partial u_x & \partial F/\partial u_y & \partial F/\partial v_x & \partial F/\partial v_y \\ \partial G/\partial u_x & \partial G/\partial u_y & \partial G/\partial v_x & \partial G/\partial v_y \end{pmatrix} = 2$$

with two unknown functions $u = u^1(x,y)$, $v = u^2(x,y)$. This is really a determined system, the solution depends on the choice of one independent variable. Assuming, e. g., $\partial F/\partial u_x \neq 0$ for certainty, then the composition series $\Omega^0 \subset \Omega^1 = \Omega$ is nontrivial if and only if the original system can be adapted as

$$\Omega = \Omega^1 \,:\, u_x - A - Cv_x = u_y - B - Cv_y = 0\,;$$

here $A, B, C$ are functions of $x, y, u, v$ such that

$$A_y - B_x + A_u B - B_u A = C_y - B_v + C_u B - B_u C = A_v - C_x + A_u C - C_u A = 0\,.$$

In this case, the differential form

$$\mathrm{d}u - C\,\mathrm{d}v - A\,\mathrm{d}x - B\,\mathrm{d}y = \mu\,\mathrm{d}f \quad (f = f(x, y, u, v))$$

is a multiple of total differential of a function $f$ such that $f \equiv$ const on every solution of $\Omega$. The subsystem $\Omega^0 \subset \Omega$ is represented by the Pfaffian equation $\mathrm{d}f = 0$ and it can be expressed by the overdetermined system

$$\Omega^0 \,:\, \partial\overline{u}/\partial x = A(x, y, \overline{u}, v)\,, \quad \partial\overline{u}/\partial y = B(x, y, \overline{u}, v)\,, \quad \partial\overline{u}/\partial v = C(x, y, \overline{u}, v)$$

for the unknown function $\overline{u} = \overline{u}(x, y, v)$. In classical terms, for every solution $u, v$ of the original system $\Omega$ there exists (in general not unique) solution $\overline{u}$ of $\Omega^0$ such that $u(x,y) = \overline{u}(x, y, v(x,y))$.

2. The second example concerns the underdetermined system of a single differential equation

$$\Omega = \Omega^2 \,:\, \partial v/\partial y = f\left(\partial u/\partial x, \partial u/\partial y, \partial v/\partial x\right)$$

with two unknown functions $u, v$. We have the underdetermined case since the function $u = u(x,y)$ can be arbitrarily chosen in advance. The underdetermined system $\Omega^0 \subset \Omega$ is always trivial, and, as a rule, the maximal determined system $\Omega^1 \subset \Omega^2$ is trivial, too. However in the exceptional case when

$$f = G\left(\partial u/\partial x, \partial v/\partial x\right) \partial u/\partial y + H\left(\partial u/\partial x, \partial v/\partial x\right)\,,$$

$$H = h(G)\,, \quad \partial v/\partial x - G\partial u/\partial x = g(G)$$

for appropriate functions $g, h, G, H$, the system $\Omega^1$ is nontrivial and consists of the equations

$$\Omega^1 : \partial\overline{v}/\partial x = g\,(\partial\overline{v}/\partial u)\ , \quad \partial\overline{v}/\partial y = h\,(\partial\overline{v}/\partial u)$$

for the unknown function $\overline{v} = \overline{v}(x, y, u)$. In classical terms, this result reads: every solution $u = u(x, y)\,, v = v(x, y)$ of the original system $\Omega^2$ satisfies the identity $v(x, y) = \overline{v}(x, y, u(x, y))$ for appropriate (not unique) solution $\overline{v}$ of $\Omega^1$.

## References

1. Chrastina J., Tryhuk V., *On the Structure of Differential Equations*, to appear.
2. Tryhuk V., Chrastinová V., *On the Mayer Problem*, to appear.
3. Chrastina J., *Examples from the Calculus of Variations I, II, III, IV*, Mathematica Bohemica 2000, 2001.
4. Chrastina J., *The Formal Theory of Differential Equations*, Folia Fac. Scient. Nat. Univ. Masarykianae Brunensis, Mathematica 6, 1998.
5. Lenin V. I., *Krok vpřed – dva kroky vzad*, VUML Heritage Publishers, 2000.