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Infinitely Many Solitary Waves in Three Space Dimensions*

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Abstract. In this paper we give some existence results obtained by V. Benci, P. d’Avenia, D. Fortunato, A. Masiello and L. Pisani about a model of Lorentz-invariant nonlinear field equation in three space dimensions which gives rise to topological solitary waves.

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1 Introduction

A *solitary wave* is a solution of a *wave equation* whose energy is finite and travels as a localized packet; a *soliton* is a solitary wave which preserves its shape after interaction, having so a *particle-like* behavior.

The soliton solutions occur in many questions of mathematical physics (non-linear optics, classical and quantum field theory, plasma physics), chemistry and biology (see [7,8,9,10]).

For some equations, the existence of soliton solutions is guaranteed by topological constraints.

In one space dimension, the simplest example of topological solitons is given by the sine-Gordon equation

$$\psi_{tt} - \psi_{xx} + \sin \psi = 0. \tag{1}$$

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If we look for finite-energy solutions, we put the asymptotic conditions

$$\begin{cases} \psi(-\infty, t) = 2h\pi \\ \psi(+\infty, t) = 2k\pi \end{cases} \quad h, k \in \mathbb{Z}$$

and just the difference $h - k$ represents the topological constraint.

It is well known that the only static solutions of the sine-Gordon equation

$$\begin{aligned} u_K(x) &= 4 \arctan(e^{x+a}) + 2h\pi \\ u_{AK}(x) &= 4 \arctan(e^{-x+a'}) + 2h'\pi \end{aligned}$$

give rise to one-soliton solutions.

Indeed, since the sine-Gordon equation is Lorentz-invariant, we consider the “travelling” solutions

$$\begin{aligned} \psi_K(x, t) &= u_K\left(\frac{x - vt}{\sqrt{1 - v^2}}\right) \\ \psi_{AK}(x, t) &= u_{AK}\left(\frac{x - vt}{\sqrt{1 - v^2}}\right) \end{aligned}$$

with $v \in \mathbb{R}$, $|v| < 1$.

Moreover, there exist other solutions of (1) which represent the *superposition* of these basic solutions.

This kind of results lead Derrick to look for stable, time-independent, localized solutions of the nonlinear wave equation

$$\psi_{tt} - \Delta\psi + V'(\psi) = 0 \tag{2}$$

in three space dimensions.

In [6] he proves that the corresponding static equation

$$-\Delta u + V'(u) = 0$$

has not finite-energy stable solutions.

In the same paper he proposes several ways to avoid this difficulty.

One of them is the following.

The equation (2) is the Euler-Lagrange equation related to the action

$$\mathcal{S}_1 = \iint \mathcal{L}_1 dx dt$$

where,

$$\mathcal{L}_1 = -\frac{1}{2}\sigma - V(\psi),$$

being

$$\sigma = |\nabla\psi|^2 - |\psi_t|^2.$$

Derrick suggests to take a Lagrangian density

$$\mathcal{L}_1 = -\alpha(\sigma) - V(\psi)$$

which gives rise to Lorentz-invariant equation, but he concludes that such kind of Lagrangian density lead to a very complicated differential equation. Indeed, in the 60ties, the methods of Nonlinear Analysis were not sufficiently developed to face quasilinear equations.

In this review, we recall some existence results concerning a nonlinear wave equation which is similar to the one proposed by Derrick.

2 The model

Let us consider the internal parameter space

$$\mathcal{M} = \mathbb{R}^4 \setminus \{\bar{\xi}\}$$

where

$$\bar{\xi} = (1, 0, 0, 0).$$

The topological solitary waves introduced in [3] are fields

$$\psi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathcal{M}.$$

The authors of [3] take the Lagrangian density

$$\mathcal{L}_1 = -\frac{1}{2} \left(\sigma + \frac{\varepsilon}{3} \sigma^3 \right) - V(\psi)$$

where $\varepsilon > 0$ and

$$V : \mathcal{M} \rightarrow \mathbb{R}.$$

We notice that this Lagrangian density is the one related to (2) plus the *correction term*

$$-\frac{\varepsilon}{6} \sigma^3.$$

The Euler-Lagrange equation related to the action

$$\mathcal{S}_1 = \iint \mathcal{L}_1 dxdt$$

is

$$\frac{\partial}{\partial t} \left((1 + \varepsilon \sigma^2) \psi_t \right) - \nabla \left((1 + \varepsilon \sigma^2) \nabla \psi \right) + V'(\psi) = 0. \tag{3}$$

Since \mathcal{M} has non-trivial topology ($\pi_3(\mathcal{M}) = \mathbb{Z}$), if we suppose that the fields ψ are smooth and

$$\lim_{|x| \rightarrow \infty} \psi(x, t) = 0, \tag{4}$$

we can classify them in the following way.

Definition 1. For every $t \in \mathbb{R}$, the topological charge of $\psi(\cdot, t)$ is

$$\text{ch}(\psi(\cdot, t)) = \deg((P \circ \psi)(\cdot, t), K(\psi(\cdot, t)), N)$$

where

$$P : \xi \mapsto |\bar{\xi}| \frac{\xi - \bar{\xi}}{|\xi - \bar{\xi}|} + \bar{\xi}$$

is the projection into the unitary sphere centered in $\bar{\xi}$,

$$N = 2\bar{\xi}$$

is the north pole of this sphere and

$$K(\psi(\cdot, t)) = \{x \in \mathbb{R}^3 \mid |\psi(\cdot, t)(x)| > 1\}.$$

If (4) is uniform with respect to t , the charge does not depend on t .

The existence results of topological solitary waves can be generalized to an arbitrary number of space dimensions with a more general choice of Lagrangian density as in [1]. The three space dimensional case is necessary to interpret the fields ψ as charged relativistic particles.

3 The static solutions

The static solutions $u = u(x)$ of (3) solve

$$-\nabla \left((1 + \varepsilon |\nabla u|^4) \nabla u \right) + V'(u) = 0$$

or, briefly,

$$-\Delta u - \varepsilon \Delta_6 u + V'(u) = 0. \tag{5}$$

Assume that

- (V1) $V \in C^2(\mathcal{M}, \mathbb{R})$;
- (V2) $V(\xi) \geq V(0) = 0$ and the Hessian matrix $V''(0)$ is non-degenerate;
- (V3) there exist $c, r > 0$ such that

$$|\xi| < r \Rightarrow V(\xi + \bar{\xi}) \geq c|\xi|^{-6}.$$

We can obtain solutions of (5) looking for critical points of the functional

$$E(u) = \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla u|^2 + \frac{\varepsilon}{6} |\nabla u|^6 + V(u) \right] dx.$$

In order to get

$$\int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla u|^2 + \frac{\varepsilon}{6} |\nabla u|^6 \right] dx < +\infty,$$

we consider the Banach space

$$H = H^1(\mathbb{R}^3, \mathbb{R}^4) \cap W^{1,6}(\mathbb{R}^3, \mathbb{R}^4).$$

By Sobolev embedding theorems we can say that the elements of H are continuous functions which go to zero at infinity.

For every u in the open subset

$$\Lambda = \{u \in H \mid \forall x \in \mathbb{R}^3 : u(x) \neq \bar{\xi}\}$$

the topological charge is well defined and the condition (V3) implies that if $u \notin \Lambda$, then

$$E(u) = +\infty.$$

Although the functional E is weakly lower semicontinuous, we cannot minimize it in the connected components

$$\Lambda_K = \{u \in \Lambda \mid \text{ch}(u) = K\}$$

since the domain \mathbb{R}^3 is not compact and Λ_K are not weakly closed.

The first existence result can be stated as follows (Theorem 2.2 of [3]).

Theorem 2. *If V satisfies (V1), (V2), (V3), then there exists a weak solution of (5) obtained as minimum of the functional E in*

$$\Lambda^* = \{u \in \Lambda \mid \text{ch}(u) \neq 0\}.$$

This result is proved by using a *Splitting Lemma* in the spirit of concentration-compactness principle.

Moreover we have the following result.

Theorem 3. *If V satisfies (V1), (V2), (V3) and for every $g \in O(3)$, and $\xi = (\xi^0, \xi^1, \xi^2, \xi^3) \in \mathcal{M}$*

$$V(\xi^0, g \cdot (\xi^1, \xi^2, \xi^3)) = V(\xi^0, \xi^1, \xi^2, \xi^3),$$

then for every $N \in \mathbb{Z}$ there exists u_N non-trivial solution of (5) such that $\text{ch}(u_N) = N$.

For $N \neq 0$, the existence of a non-trivial solution is proved in [1]. The authors use a suitable invariance of the functional E in order to avoid the lack of compactness.

For $N = 0$ the existence of a nontrivial solution is obtained in [5] using the Hopf invariant.

A function which satisfies all these conditions is

$$V(\xi) = \omega_0^2 \left(|\xi|^2 + \frac{|\xi|^4}{|\xi - \bar{\xi}|^6} \right).$$

4 Interaction with the electromagnetic field

4.1 Charge density and current density

If we want to interpret the fields ψ as charged relativistic particles, it is natural to attribute to the topological charge the meaning of electric charge.

In [2], the authors introduce the fields

$$\begin{aligned}\mathbf{J}(\psi) &: \mathbb{R}^4 \rightarrow \mathbb{R}^3 \\ \rho(\psi) &: \mathbb{R}^4 \rightarrow \mathbb{R}\end{aligned}$$

with the meaning of charge density and current density generated by ψ .

Let

$$\eta = \sum_{k=0}^3 \eta_k(\xi) d\xi^0 \wedge \dots \wedge \widehat{d\xi^k} \wedge \dots \wedge d\xi^3$$

be the unique 3-form closed but not exact on \mathcal{M} , where the hatted symbols are omitted,

$$\eta_k(\xi) = \frac{1}{|\Sigma|} \frac{(-1)^k (\xi^k - \bar{\xi}^k)}{|\xi - \bar{\xi}|^4}$$

and $|\Sigma|$ is the measure of the unitary sphere in \mathbb{R}^4 .

Let $\psi^*\eta$ denote the pullback of η by ψ .

The Hodge operator applied to $\psi^*\eta$ gives a 1-form on \mathbb{R}^4

$$*(\psi^*\eta). \tag{6}$$

The fields (\mathbf{J}, ρ) introduced in [2] are the components of (6).

Indeed we can verify that

$$\text{ch}(\psi(\cdot, t)) = \int_{\mathbb{R}^3} \rho(\psi) dx$$

(see Appendix of [4]).

On the other hand, since η is closed, the pullback $\psi^*\eta$ is closed and then

$$d(\psi^*\eta) = 0.$$

This can be written as the continuity equation

$$\nabla \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

Remark 4. When we consider static fields $u = u(x)$, since in the expression of $J_i(\psi)$ appears the factor $\frac{\partial \psi^j}{\partial t}$, we have

$$\mathbf{J} = \mathbf{0},$$

namely, as it is natural, in this case there is not electric current.

4.2 The system solitary wave-e.m. field

Let (\mathbf{A}, ϕ) denote the gauge potential associated to the electromagnetic field.

Using (\mathbf{A}, ϕ) , we can define the electric field \mathbf{E} and the magnetic induction field \mathbf{B} as follows

$$\mathbf{E} = -(\mathbf{A}_t + \nabla\phi) \tag{7}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{8}$$

By (7) and (8) we get immediately the first two Maxwell equations

$$\nabla \times \mathbf{E} + \mathbf{B}_t = 0$$

$$\nabla \cdot \mathbf{B} = 0.$$

In the vacuum, with a suitable choice of the physical constants, the second pair of the Maxwell equations

$$\nabla\mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{B} - \mathbf{E}_t = 4\pi\mathbf{J}$$

can be obtained as the Euler-Lagrange equations related to the action

$$\mathcal{S}_{emf} = \iint (\mathcal{L}_2 + \mathcal{L}_3) dxdt$$

where

$$\mathcal{L}_2 = \frac{1}{8\pi} (|\mathbf{E}|^2 - |\mathbf{B}|^2)$$

is the Lagrangian density of the electromagnetic field and

$$\mathcal{L}_3 = (\mathbf{J}|\mathbf{A}) - \rho\phi.$$

Hence, if we consider the system solitary wave-electromagnetic field, the total action is

$$\mathcal{S} = \mathcal{S}(\psi, \mathbf{A}, \phi) = \mathcal{S}_1(\psi) + \mathcal{S}_{emf}(\psi, \mathbf{A}, \phi).$$

The Euler-Lagrange equations in the static case are

$$-\Delta u - \varepsilon\Delta_\delta u + V'(u) = G \tag{9}$$

$$\nabla \times (\nabla \times \mathbf{A}) = 0 \tag{10}$$

$$-\Delta\phi = 4\pi\rho(u) \tag{11}$$

where G derives from the interaction term and depends on u, ϕ (and their derivatives).

We have the following results:

Theorem 5. *If V satisfies (V1), (V2), (V3), then there exist*

$$u \in H \text{ and } \phi \in D^{1,2}(\mathbb{R}^3, \mathbb{R})$$

such that

- $\text{ch}(u) \neq 0$;
- $(u, 0, \phi)$ is solution of (9, 10, 11).

(Theorem 1.1 of [2])

Theorem 6. If V satisfies (V1), (V2), (V3), and for every $g \in O(3)$, and $\xi = (\xi^0, \xi^1, \xi^2, \xi^3) \in \mathcal{M}$

$$V(\xi^0, g \cdot (\xi^1, \xi^2, \xi^3)) = V(\xi^0, \xi^1, \xi^2, \xi^3),$$

then for every $N \in \mathbb{Z}$ there exist

$$u_N \in H \text{ and } \phi_N \in D^{1,2}(\mathbb{R}^3, \mathbb{R})$$

such that

- $\text{ch}(u_N) = N$;
- $(u_N, 0, \phi_N)$ is a non-trivial solution of (9, 10, 11).

(Theorem 7 of [4])

The static solutions of (9, 10, 11) are obtained as critical points of the functional

$$f(u, \phi) = \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla u|^2 + \frac{\varepsilon}{6} |\nabla u|^6 + V(u) \right) dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx + \int_{\mathbb{R}^3} \phi \rho(u) dx$$

which is strongly indefinite.

Then the solutions are obtained using the *reduced* functional

$$J(u) = f(u, \Phi[u])$$

where $\Phi[u]$ is implicitly defined by

$$\frac{\partial f}{\partial \phi} = 0.$$

Remark 7. If \bar{u} is the non-trivial static solution of (5) having $\text{ch}(\bar{u}) = 0$ found in Theorem 2 of [5], an immediate calculation shows that $(\bar{u}, 0, 0)$ is a solution of (9, 10, 11) (Theorem 4 of [5]).

This is a further confirmation of the model's coherence: indeed an uncharged particle does not create any electromagnetic field.

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