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Positive and oscillating solutions of equation $\dot{x}(t) = -c(t)x(t - \tau)$


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Positive and oscillating solutions of equation
\[ \dot{x}(t) = -c(t)x(t - \tau) \]

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Abstract. Positive and oscillating solutions of delayed equation
\[ \dot{x}(t) = -c(t)x(t - \tau) \]
with \( c \in C(I, \mathbb{R}^+) \), \( I = [t_0, \infty) \), \( \mathbb{R}^+ = (0, \infty) \) and \( 0 < \tau = \text{const} \) are studied.

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Let us consider the equation
\[ \dot{x}(t) = -c(t)x(t - \tau) \] (1)
where \( c \in C(I, \mathbb{R}^+) \), \( I = [t_0, \infty) \), \( \mathbb{R}^+ = (0, \infty) \) and \( 0 < \tau = \text{const} \).

Define \( \ln_k t = \ln(\ln_{k-1} t), k \geq 1 \) where \( \ln_0 t \equiv t \) for \( t > \exp_k 1 \) where \( \exp_k t \equiv (\exp(\exp_{k-1} t)), k \geq 1, \exp_0 t \equiv t \) and \( \exp_{-1} t \equiv 0 \). (Instead of expressions \( \ln_0 t, \ln_1 t \) is only \( t \) and \( \ln t \) written in the sequel.) Moreover, define so called critical functions for (1)
\[ c_k(t) \equiv \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t \ln t)^2} + \frac{\tau}{8e(t \ln t \ln_2 t)^2} + \cdots + \frac{\tau}{8e(t \ln t \ln_2 t \ldots \ln_k t)^2} \]
with \( k \geq 0 \).

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This is the preliminary version of the paper.
Theorem 1. [1]
A) Let us assume that \(c(t) \leq c_k(t)\) for \(t \to \infty\) and an integer \(k \geq 0\). Then there is a positive solution \(x = x(t)\) of Eq. (1). Moreover,
\[x(t) < \nu_k(t) \equiv e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_k t}\]
as \(t \to \infty\).

B) Let us assume that
\[c(t) > c_{k-1}(t) + \frac{\theta \tau}{8e(t \ln t \ln_2 t \cdots \ln_k t)^2}\]
for \(t \to \infty\), an integer \(k \geq 1\) and a constant \(\theta > 1\). Then all solutions of Eq. (1) oscillate.

Theorem 2. [1] Assume that the inequality (2) holds for \(t \to \infty\), an integer \(k \geq 1\) and a constant \(\theta > 1\). Then each solution of Eq. (1) has at least one zero on each interval \((p - \tau, q)\) for \(q = \exp_k(\ln_k p)\exp(\pi/\zeta), \zeta^2 < (\theta - 1)/4\), (\(\zeta\) is a positive constant) and \(p\) sufficiently large.

Theorem 3. [2] Let there exists a positive solution \(\tilde{x}\) of (1) on \(I\). Then there are positive solutions \(x_1\) and \(x_2\) of (1) on \(I\) satisfying the relation
\[\lim_{t \to \infty} \frac{x_2(t)}{x_1(t)} = 0.\]
Moreover, every solution \(x\) of (1) on \(I\) is represented by the formula
\[x(t) = Kx_1(t) + O(x_2(t))\]
where \(K \in \mathbb{R}\) depends on \(x\).

Definition 4. [3] Let \(x_1\) and \(x_2\) be fixed positive solutions of the delayed equation (1) on \(I\), with the property (3). Then \((x_1, x_2)\) is called a pair of dominant and subdominant solutions on \(I\).

Let us consider the equation (1) in the case when the coefficient \(c\) is equal to a critical function, i.e., in the case of equation
\[\dot{x}(t) = -c_k(t)x(t - \tau), \quad k \geq 0; \quad t \geq t_0 > \exp_{p-1}.\]

Theorem 5. [3] Let \(k \geq 0\) be fixed. Then for any fixed constants \(\delta_1 > 2\) and \(\delta_2 < 0\) there are a \(t_0\), and a pair \((x_1, x_2)\) of dominant and subdominant solutions of (4) on \(I\) satisfying the two-sided estimates
\[e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1}^2 t} < x_1(t) < e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1}^5 t}\]
and
\[e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1}^{\delta_2} t} < x_2(t) < e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t}\]
on \(I\).
References
