Patrick Habets; Luis Sanchez; Massimo Tarallo; Susanna Terracini

Heteroclinics for a class of fourth order conservative differential equations

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 203--216.

Persistent URL: http://dml.cz/dmlcz/700354

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 203–216

Heteroclinics for a class of fourth order conservative differential equations

P. Habets[1], L. Sanchez[2⋆], M. Tarallo[3] and S. Terracini[4]

[1] Institut de Mathématique Pure et Appliquée,
Chemin du Cyclotron, 2, 1348 Louvain la Neuve, Belgium,
Email: habets@anma.ucl.ac.be

[2] Universidade de Lisboa, Centro de Matemática e Aplicações Fundamentais,
Avenida Professor Gama Pinto, 2, 1649-003 Lisboa, Portugal,
Email: sanchez@lmc.fc.ul.pt

[3] Università degli studi di Milano, Via Saldini 50, Milano, Italy,
Email: massimo.tarallo@mat.unimi.it

[4] Politecnico di Milano, Via Bonardi 9, Milano, Italy,
Email: susterra@tin.it

Abstract. We prove the existence of heteroclinics for a 4th order O.D.E. related to the extended Fisher-Kolmogorov equation. Those solutions are obtained by minimization of a functional over a convenient set of functions. In particular, we
obtain heteroclinic connections between the extreme equilibria for a (double well) potential with three minima at the same level.

MSC 2000. 34B15, 34C37

Keywords. heteroclinic, double well potential

1 Introduction

In the study of ternary mixtures containing oil, water and amphiphile, a modification of a Ginzburg-Landau model yields for the free energy a functional of the form (see[2])

$$\mathcal{F}(u) = \int [c(\nabla^2 u)^2 + g(u)|\nabla u|^2 + f(u)]\, dx\, dy\, dz$$

⋆ Research supported by Fundação para a Ciência e a Tecnologia.

This is the preliminary version of the paper.

where the scalar order parameter u is related to the local difference of concentrations of water and oil. The function $g(u)$ quantifies the amphiphilic properties and the "potential" $f(u)$ is the bulk free energy of the ternary mixture. In some relevant situations g may take negative values to an extent that is balanced by the positivity of c and f.

The admissible density profiles may therefore be identified with critical points of $\mathcal{F}$ in a suitable function space. In the simplest case where the order parameter depends only on one spatial direction, $u = u(x)$ is defined on the real line and (after scaling) our functional becomes

$$\mathcal{F}(u) = \int_{-\infty}^{+\infty} [\frac{1}{2}[(u''^2) + g(u)u'^2] + f(u)]\, dx. \tag{1}$$

The corresponding Euler-Lagrange equation is

$$u^{iv} - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{2}$$

When $g \equiv const = \beta$, we recognize here the well known extended Fisher-Kolmogorov equation. If the potential $f(u)$ has two nondegenerate minima, say, at ± 1, with $f(\pm 1) = 0$, one question of great interest is the existence of a heteroclinic that connects these two equilibria. Our purpose is to obtain such a solution to (2) in cases where g may change sign. However, we shall be interested also in the case where f possesses three minima at the same level, since this is the framework where description of the three distinct phases becomes possible (see [2,4,5]).

Let us recall that the case $g \equiv const = \beta \neq 0$, has attracted the attention of several authors. Peletier and Troy (see [7] and related papers in its references) have extensively dealt with the case $\beta > 0$ and the model potential $f(u) = \frac{1}{4}(u^2 - 1)^2$; they have shown that a heteroclinic connecting ± 1 exists for all values of $\beta > 0$. The cases $\beta^2 < 8$ ("saddle-focus" case) and $\beta^2 \geq 8$ ("saddle-node" case) need a different treatment. Kalies and VanderVorst [6] have considered an even potential in the saddle-foci case. In [3] Kalies, Kwaspisz and VanderVorst have classified heteroclinic connections (with $\beta > 0$) between the two consecutive equilibria according to their homotopy type. Jan Bouwe van den Berg [1] has proved that if $\beta^2 \geq 8$ the heteroclinic is asymptotically stable. More recently Smets and van den Berg have considered the case $\beta < 0$ for which they prove (in a saddle-foci case), by a version of the mountain pass theorem, that at each equilibria there arise homoclinic solutions.

Here we start by considering the simple case $g \equiv 0$ and address two problems that, however, seem not to be covered by the existing literature. First (section 2) we prove the existence of the heteroclinic without the assumption of symmetry for f. Second (section 3), we look at the case where the potential f is symmetric, having not two, but three equilibria (0 and ± 1) at the same minimum level: we shall see that a heteroclinic connecting ± 1 still exists in this case.

For simplicity, we deal with a potential that reduces to a quadratic function near ± 1, see assumption (F1). However, by using the Hartmann-Grobman theorem, we could instead consider any nondegenerate minima at these points.

Of course, in the case $g \equiv \beta = 0$ our functional becomes

$$\mathcal{J}(u) = \int_{-\infty}^{+\infty} [\frac{1}{2}(u''^2) + f(u)]\, dx. \tag{3}$$

and its corresponding Euler-lagrange equation is

$$u^{iv} + f'(u) = 0. \tag{4}$$

Finally, in section 4 we shall introduce a "compatibility condition" relating g and f in order to consider the original problem (1), (2). That condition allows g to take negative values somewhere between ± 1 as required in the the theory of ternary mixtures; it is consistent with physical arguments (see [2]) and from the mathematical point of view, it enables us to reduce the problem to the simpler case $g \equiv 0$, since we are then able to construct a functional similar to $\mathcal{J}$ that bounds $\mathcal{F}$ from below.

A variety of numerical results leading to solutions of equations of type (2), obtained by minimization of the free energy $\mathcal{F}$, can be found in the thesis of H. Leitão [4].

The authors are indebted to H. Leitão for having brought this problem to their attention. We aknowledge also the interest and useful discussions with D. Bonheure.

2 Potentials with a single well

We consider a potential $f \in C^2(\mathbb{R})$ such that for some $0 < a < 1/2$ and $\alpha > 0$,

$(F1)$
$$f(u) = 2\alpha^4(u-1)^2, \quad \forall u \in (1-a, 1+a),$$
$$f(u) = 2\alpha^4(u+1)^2, \quad \forall u \in (-1-a, -1+a),$$

$(F2)$
$$f(u) = 0 \text{ if and only if } u = \pm 1$$

and

$(F3)$
$$\liminf_{|u|\to\infty} f(u) > 0.$$

Lemma 1. *Given an interval $[a, b] \subset \mathbb{R}$ and a function $u \in H^2(a, b)$ such that $u(a) = A,\ u(b) = B,\ u'(a) = A_1,\ u'(b) = B_1$ the following inequality holds:*

$$\int_a^b u''^2\, dx \geq \frac{4}{b-a}[(B_1 - A_1)^2 + 3(\frac{B-A}{b-a} - A_1)(\frac{B-A}{b-a} - B_1)]$$

and equality holds if and only if u is a 3rd degree polynomial.

We introduce the space $\mathcal{E}$ consisting of all functions u defined in $\mathbb{R}$ such that $u \in C^1(\mathbb{R})$, $u'' \in L^2(\mathbb{R})$ and

$$\lim_{x\to\pm\infty} u(x) = \pm 1.$$

In order to minimize $\mathcal{J}$ in $\mathcal{E}$ we shall now obtain estimates on functions $u \in \mathcal{E}$ in terms of an upper bound of the values $\mathcal{J}(u)$.

Lemma 2. *There are constants K and N, depending only on C, such that, for any function $u \in \mathcal{E}$, $\mathcal{J}(u) \le C$ implies*

$$\|u\|_\infty \le K, \quad \|u'\|_\infty \le N.$$

Proof. According to the assumptions on f there exist $K > 0$ and $A > 0$ such that

$$\frac{K^2}{A^3} > C \quad \text{and} \quad A\,f(u) > C \quad \forall |u| \ge \frac{K}{2}.$$

We claim that $\|u\|_\infty \le K$. Otherwise, either the set $\{x : |u(x)| \ge K/2\}$ has measure greater than A and

$$\mathcal{J}(u) \ge \int_{-\infty}^{+\infty} f(u)\,dx > A\frac{C}{A} = C,$$

a contradiction, or we can pick up an interval (c,d) such that $d - c < A$, $u(c) = |u|_\infty$, $u(d) = \frac{K}{2}$, $u(x) \ge \frac{K}{2}$, $\forall x \in (c,d)$ and therefore by lemma 1

$$\mathcal{J}(u) \ge \int_c^d \frac{u''^2}{2}\,dx \ge \frac{2}{d-c}[u'(d)^2 + 3(\frac{u(d)-u(c)}{d-c} - u'(d))(\frac{u(d)-u(c)}{d-c})]$$

$$\ge \frac{4}{d-c}[\frac{u(d)-u(c)}{d-c}]^2 \ge \frac{K^2}{A^3} > C,$$

again a contradiction. Hence the first part of the statement is proved.

Next choose N such that

$$N > 4K, \quad , N^2 > 8C.$$

We show that u' cannot attain the value N and one shows analogously that it cannot attain the value $-N$. For suppose that $u'(t_0) = N$; then there exists $t_1 \in (t_0, t_0+1)$ such that $u'(t_1) = \frac{N}{2}$. Hence, letting $m = \frac{u(t_1)-u(t_0)}{t_1-t_0}$ it turns out that

$$\mathcal{J}(u) \ge \int_{t_0}^{t_1} \frac{u''^2}{2}\,dx \ge \frac{2}{t_1-t_0}[(\frac{N}{2})^2 + 3(m-N)(m-\frac{N}{2})] \ge \frac{N^2}{8} > C,$$

which is impossible.

Lemma 3. *If* $u \in \mathcal{E}$ *and* $\mathcal{J}(u) < \infty$ *then*

$$\lim_{|x| \to \infty} u'(x) = 0.$$

Proof. This is a variation on the above argument. If, for instance, there exists $\epsilon > 0$ and a sequence $x_n \to +\infty$ with $u'(x_n) \geq \epsilon$ for all n then, since $u(+\infty) = 1$, $\forall \delta > 0$ we can pick up an interval (t_0, t_1) such that $u'(t_0) \geq \epsilon$, $u'(t_1) = \frac{\epsilon}{2}$, $u(t_1) \leq u(t_0) + \delta$ and $t_1 - t_0 < \frac{2\delta}{\epsilon}$. As before, we derive (with the same meaning for m)

$$2\mathcal{J}(u) \geq \int_{t_0}^{t_1} u''^2 \, dx \geq \frac{4\epsilon}{2\delta}[(\frac{\epsilon}{2})^2 + 3(m - \epsilon)(m - \frac{\epsilon}{2})] \geq \frac{\epsilon^3}{8\delta}.$$

The main idea in the next lemma is that there is an upper bound, depending only on the value of $\mathcal{J}$, for the time it takes for a function $u \in \mathcal{E}$ to travel in the (u, u')-plane from a neighborhood of $(-1, 0)$ to a neighborhood of $(1, 0)$.

Lemma 4. *Let* $C > 0$ *and* $\epsilon > 0$ *be given. Then there exists* $R > 0$ *such that for any function* $u \in \mathcal{E}$ *with* $\mathcal{J}(u) \leq C$*, there exist* x_1 *and* $x_2 > x_1$ *that satisfy*

$$|u(x_i) - (-1)^i| \leq \epsilon, \quad |u'(x_i)| \leq \epsilon \quad \text{and} \quad x_2 - x_1 \leq R.$$

Proof. Let $C > 0$ and $\epsilon > 0$ be given and let $u \in \mathcal{E}$ be so that $\mathcal{J}(u) \leq C$. Define $x_1 = \sup\{x \mid |u(x) + 1| \leq \epsilon \text{ and } |u'(x)| \leq \epsilon\}$. As $u \in \mathcal{E}$, it is clear that $x_1 \in \mathbb{R}$. Now given $x_2 > x_1$ suppose that

$$\forall x \in [x_1, x_2], \quad |u(x) - 1| \geq \epsilon \ \text{ or } \ |u'(x)| \geq \epsilon. \tag{5}$$

We shall give a bound on $x_2 - x_1$ in terms of C and ϵ.

Define the sets

$$A = \{x \in [x_1, x_2] \mid |u(x) + 1| \geq \epsilon \ \text{ and } \ |u(x) - 1| \geq \epsilon\},$$

and

$$B = \{x \in [x_1, x_2] \mid |u(x) + 1| < \epsilon \ \text{ or } \ |u(x) - 1| < \epsilon\}.$$

It is easy to see that B is the union of intervals I_i on which $|u'(x)| \geq \epsilon$. Further except maybe for the first and the last one, each of these intervals is adjacent to an interval $J_i = [c_i, d_i]$ so that

$$\forall x \in [c_i, d_i], \ u(x) \geq 1 + \epsilon, \quad u'(c_i) \geq \epsilon, \ u'(d_i) \leq -\epsilon,$$

or

$$\forall x \in [c_i, d_i], \ u(x) \leq -1 - \epsilon, \quad u'(c_i) \leq -\epsilon, \ u'(d_i) \geq \epsilon.$$

Claim 1 – $\text{meas}(A) \leq \frac{C}{r_\epsilon}$, *where*

$$r_\epsilon = \min\{f(u) \mid |u + 1| \geq \epsilon \ \text{ and } \ |u - 1| \geq \epsilon\}.$$

This follows from the inequalities

$$C \geq \mathcal{J}(u) \geq \int_A f(u(x)) \, dx \geq r_\epsilon \text{meas}(A).$$

Claim 2 – $\mathrm{meas}(I_i) \le 2$. On an interval $\bar{I}_i = [a_i, b_i]$, we have $|u'(x)| \ge \epsilon$ and

$$2\epsilon \ge |u(b_i) - u(a_i)| = |\int_{a_i}^{b_i} u'(x)\,dx| \ge \epsilon(b_i - a_i).$$

Claim 3 – The number n of intervals J_i is bounded : $n \le C/\min\{2\epsilon^2, r_\epsilon\}$. Let $J_i = [c_i, d_i]$ be such that $\forall t \in [c_i, d_i]$, $u(t) \ge 1 + \epsilon$, $u'(c_i) \ge \epsilon$ and $u'(d_i) \le -\epsilon$. We can write

$$2\epsilon \le |u'(d_i) - u'(c_i)| = |\int_{c_i}^{d_i} u''(x)\,dx| \le \|u''\|_{L^2(c_i,d_i)}(d_i - c_i)^{1/2}.$$

and

$$\int_{c_i}^{d_i} [\tfrac{1}{2}(u'')^2 + f(u)]\,dx \ge \frac{2\epsilon^2}{d_i - c_i} + r_\epsilon(d_i - c_i) \ge \min\{2\epsilon^2, r_\epsilon\}.$$

A similar argument holds if $\forall x \in [c_i, d_i]$, $u(x) \le -1-\epsilon$, $u'(c_i) \le -\epsilon$ and $u'(d_i) \ge \epsilon$. It follows then that

$$C \ge \mathcal{J}(u) \ge \sum_i \int_{c_i}^{d_i} [\tfrac{1}{2}(u'')^2 + f(u)]\,dx \ge n \min\{2\epsilon^2, r_\epsilon\}.$$

Conclusion – We deduce from the previous claims that

$$x_2 - x_1 = \ \mathrm{meas}(A) + \ \mathrm{meas}(B) \le \tfrac{C}{r_\epsilon} + (C/\min\{2\epsilon^2, r_\epsilon\} + 2)2$$

and the lemma follows.

For convenience we introduce the notation

$$\mathcal{V}_+(T, \epsilon) := \{u \in \mathcal{E} : \, |u(t) - 1| \le \epsilon, \; |u'(t)| \le \epsilon \;\; \forall t \ge T\}$$

where T and ϵ are given positive numbers. Analogously we define $\mathcal{V}_-(T, \epsilon)$ replacing -1 for 1 and $t \le -T$ for $t \ge T$ in the above definition. Let us also set

$$\mathcal{V}(T, \epsilon) = \mathcal{V}_+(T, \epsilon) \cap \mathcal{V}_-(T, \epsilon).$$

Lemma 5. *Let $C > 0$ and $\epsilon > 0$ be given. Then there exists $R > 0$ such that for any $u \in \mathcal{E}$ with $\mathcal{J}(u) \le C$, there exists $v \in \mathcal{V}(R, \epsilon)$ that satisfies*

$$\mathcal{J}(v) \le \mathcal{J}(u).$$

Proof. Fix ϵ_1 such that

$$\epsilon_1(2 + \frac{1}{\alpha}) \le \min\{a, \epsilon\}, \quad (4\alpha + 2)\epsilon_1 \le \epsilon, \quad \text{and} \quad (\alpha + 2\alpha^2 + 2\alpha^3)\epsilon_1^2 \le \frac{\alpha^4 a^3}{4N}.$$

Let R be given as in Lemma 4 with respect to C and ϵ_1. Let $u \in \mathcal{E}$ be such that $\mathcal{J}(u) \le C$. There exist $x_1 \le x_2$ such that

$$|u(x_i) - (-1)^i| \le \epsilon_1, \quad |u'(x_i)| \le \epsilon_1 \quad \text{and} \quad x_2 - x_1 \le R.$$

Define then the function v as

$$v(x) = \begin{cases} -1 + z(x) & \text{if } x \le x_1, \\ u(x) & \text{if } x_1 \le x \le x_2, \\ 1 + w(x) & \text{if } x \ge x_2, \end{cases}$$

where z and w are respectively the solutions of

$$z^{iv} + 4\alpha^4 z = 0, \quad -1 + z(x_1) = u(x_1),\ z'(x_1) = u'(x_1),\ z(-\infty) = 0$$

and

$$w^{iv} + 4\alpha^4 w = 0, \quad 1 + w(x_2) = u(x_2),\ w'(x_2) = u'(x_2),\ w(+\infty) = 0.$$

Next we compute

$$z(x) = e^{\alpha(x-x_1)}[z(x_1)\cos\alpha(x-x_1) + (\tfrac{z'(x_1)}{\alpha} - z(x_1))\sin\alpha(x-x_1)],$$

where

$$|z(x_1)| \le \epsilon_1, \quad |z'(x_1)| \le \epsilon_1.$$

It follows that for all $x \le x_1$, we have $|z(x)| \le \epsilon_1(2 + \frac{1}{\alpha}) \le \min\{a, \epsilon\}$, $|z'(x)| \le (4\alpha + 2)\epsilon_1 \le \epsilon$ and

$$\begin{aligned} \mathcal{K}(z) &:= \int_{-\infty}^{x_1} [\tfrac{1}{2}(z'')^2 + 2\alpha^4 z^2]\, dx = \tfrac{1}{2}[z''(x_1)z'(x_1) - z'''(x_1)z(x_1)] \\ &= \alpha z'(x_1)^2 - 2\alpha^2 z(x_1)z'(x_1) + 2\alpha^3 z(x_1)^2 \le (\alpha + 2\alpha^2 + 2\alpha^3)\epsilon_1^2 < \tfrac{\alpha^4 a^3}{4N}. \end{aligned}$$

If for any $x \in]-\infty, x_1]$, $u(x) \in [-1-a, -1+a]$, we compute

$$\int_{-\infty}^{x_1} [\tfrac{1}{2}(u'')^2 + f(u)]\, dx = \mathcal{K}(1+u) \ge \mathcal{K}(z).$$

Here we used the fact that, as z is a critical point of the convex functional $\mathcal{K}(u)$, it is a minimum. On the other hand, if there exists $x \in]-\infty, x_1]$ such that $u(x) \notin [-1-a, -1+a]$, there exist $x_3 \le x_4 \le x_1$ so that

$$u(x_3) = -1 + a/2, \quad u(x_4) = -1 + a$$

and

$$\forall x \in [x_3, x_4],\ \ u(x) \in [-1 + a/2, -1 + a].$$

It follows that

$$N(x_4 - x_3) \ge \int_{x_3}^{x_4} u'(x)\, dx = \tfrac{a}{2} = u(x_4) - u(x_3) = \tfrac{a}{2}$$

and

$$\int_{-\infty}^{x_1} [\tfrac{1}{2}(u'')^2 + f(u)]\, dx \ge \int_{x_3}^{x_4} [\tfrac{1}{2}(u'')^2 + f(u)]\, dx \ge \tfrac{\alpha^4 a^3}{4N}.$$

A similar computation holds for the interval $[x_2, +\infty[$ and we can write

$$\begin{aligned}\mathcal{J}(v) &= \int_{-\infty}^{x_1} [\tfrac{1}{2}(z'')^2 + 2\alpha^4 z^2]\, dx \\ &\qquad + \int_{x_1}^{x_2} [\tfrac{1}{2}(u'')^2 + f(u)]\, dx + \int_{x_2}^{\infty} [\tfrac{1}{2}(w'')^2 + 2\alpha^4 w^2]\, dx \\ &\le \int_{-\infty}^{x_1} [\tfrac{1}{2}(u'')^2 + 2\alpha^4 (u+1)^2]\, dx \\ &\qquad + \int_{x_1}^{x_2} [\tfrac{1}{2}(u'')^2 + f(u)]\, dx + \int_{x_2}^{\infty} [\tfrac{1}{2}(u'')^2 + 2\alpha^4 (u-1)^2]\, dx \\ &= \mathcal{J}(u).\end{aligned}$$

At last, it is clear that $v \in \mathcal{C}^1(\mathbb{R})$,
$v'' \in L^2(\mathbb{R})$ and as $z(-\infty) = w(\infty) = 0$ we can write $v \in \mathcal{E}$. Translating v if necessary, we can assume $-R \le x_1 \le x_2 \le R$ so that $v \in \mathcal{V}(R, \epsilon)$.

Lemma 6. *Let a function $u \in C^1(\mathbb{R})$ be such that $u'' \in L^2(\mathbb{R})$, u' is bounded, $\mathcal{J}(u) < +\infty$ and there exist $R > 0$ and $0 < \epsilon < 1$ such that*

$$|u(x) - (-1)^i| < \epsilon \quad \text{if} \quad (-1)^i x > R, \quad i = 1, 2.$$

Then $u \in \mathcal{E}$.

Proof. Clearly, if u satisfies the assumptions of the lemma then because of (F1)-(F2)-(F3)

$$\liminf_{x \to +\infty} u(x) \le 1 \le \limsup_{x \to +\infty} u(x).$$

Strict inequalities are easily ruled out since u' is bounded. The statement now follows from Lemma 3.

Theorem 7. *There exists a minimizer u of $\mathcal{J}$ in $\mathcal{E}$ which is a heteroclinic of (4) connecting ± 1.*

Proof. Let $m := \inf_{u \in \mathcal{E}} \mathcal{J}(u)$ and choose a minimizing sequence $(u_n)_n \subset \mathcal{E}$ such that $\mathcal{J}(u_n) \le m + 1/n$, for any $n \in \mathbb{N}$.

Fix $0 < \epsilon < \frac{1}{2}$. According to Lemma 5, there exist $R > 0$ and a sequence $(v_n)_n \subset \mathcal{V}(R, \epsilon)$ that satisfies $\mathcal{J}(v_n) \le \mathcal{J}(u_n) \le m + \frac{1}{n}$.
Estimates in Lemma 2 imply that $(v_n)_n$ has a subsequence (still written $(v_n)_n$ for simplicity) such that for some function v

$$v_n \overset{\mathcal{C}_{loc}(\mathbb{R})}{\longrightarrow} v, \quad v_n'' \overset{L^2(\mathbb{R})}{\rightharpoonup} v''.$$

By Fatou's lemma, we have $\mathcal{J}(v) \le m$. On the other hand it is clear that $|v(x) - 1| \le \epsilon$ and $|v'(x)| \le \epsilon$ for $|x| \ge R$. It follows from Lemma 6 that $v \in \mathcal{E}$ and therefore $\mathcal{J}(v) = m$. The proof is complete.

3 Symmetric potentials with a double well

In this section we assume that f is a C^2 even function with three minima at the same level, namely we replace (F2) by the following:

$(F2')$ $\qquad f(u) = 0$ if and only if $u = 0$ or $u = \pm 1$.

and we introduce

$(F4)$ $\qquad f$ is even and increasing on some interval $]0, b[, \quad b < 1$.

Theorem 8. *Assume that f satisfies (F1),(F2'), (F3) and (F4). Then there exists an odd heteroclinic of (4), with $u'(0) > 0$, connecting ± 1.*

Proof. As in [6] we now look for a minimizer of

$$\mathcal{J}_0(u) = \int_0^{+\infty} [\frac{1}{2}(u''^2) + f(u)]\, dx.$$

in the set $\mathcal{E}_0$ consisting of functions $u \in C^1([0, +\infty))$ with $u(0) = 0$, $u(+\infty) = 1$ and $u'' \in L^2(0, +\infty)$. A minimizer will satisfy (4) in $[0, +\infty)$ and the natural boundary condition $u''(0) = 0$; hence its odd extension is the solution one looks for.

First we note that estimates for the C^1 norm of u in terms of an upper bound of $\mathcal{J}_0(u)$ immediately follow from Lemma 2. In order to construct a compact minimizing sequence in $\mathcal{E}_0$ it suffices to show that, given a small number $\epsilon > 0$, there exists $T > 0$ such that $u \in \mathcal{E}_0$ may be replaced with $v \in \mathcal{E}_0$ such that
$\mathcal{J}_0(v) \leq \mathcal{J}_0(u)$ and $v \in \mathcal{V}_+(T, \epsilon)$.

By the argument of lemma 4, given $u \in \mathcal{E}_0$ with $\mathcal{J}_0(u) < \infty$ there exists $t^* = \inf\{t > 0 : |u(t) - 1| \leq \epsilon, \quad |u'(t)| \leq \epsilon\}$. We replace the "tail" of u (the restriction of u to $t \geq t^*$) with a solution of a linear problem translated to 1 as in the proof of lemma 5. Also, using arguments similar to those appearing in the proof of lemma 4, it is easy to estimate, in terms of the value $C := \mathcal{J}_0(u)$, the length of the interval $[t', t^*]$ such that $u(t') = b/2$ and $u(t) \geq b/2$ if $t \in [t', t^*]$. Because of (F2') we can find $\gamma > 0$, depending only on C, such that $\max\{u'(t) : t \in [t', t^*],\ \ b/2 \leq u(t) \leq b\} \geq \gamma$. Let $\bar{t}$ be a point where this maximum is attained and consider the function $w \in C^1(\mathbb{R})$ defined by

$$w(x) = \begin{cases} u(x) & \text{if } x \geq \bar{t} \\ u(\bar{t}) + u'(\bar{t})(x - \bar{t}) & \text{if } x \leq \bar{t} \end{cases} \tag{7}$$

Now the zero θ of w to the left of $\bar{t}$ depends only on $u'(\bar{t})$ and therefore on C. If $w(x) \leq u(x)\ \forall x \in [\theta, \bar{t}]$ we define $v(x) = w(x + \theta)$ for $x \geq 0$. If this is not the case, then $\exists \tilde{t} \in [\theta, \bar{t}]$ such that $u'(\tilde{t}) = \max\{u'(t) : t \in [\theta, t^*],\ u(t) \geq 0\} \geq u'(\bar{t})$ and $0 \leq u(\tilde{t}) < b/2$. In this case we consider $\tilde{w}$ defined as in (7) with $\tilde{t}$ instead of $\bar{t}$. Clearly, the zero $\tilde{\theta}$ of $\tilde{w}$ is $\geq \theta$, and we define $v(x) = w(x + \tilde{\theta})$ for $x \geq 0$. In any case it is obvious, because of (F4), that $\mathcal{J}_0(v) \leq \mathcal{J}_0(u)$
and we can take $T = t^* - \theta$.

4 Back to the original problem

In this section we reconsider the functional $\mathcal{F}$ and its Euler-Lagrange equation (2). We shall assume that g is a C^1 function in $\mathbb{R}$ satisfying

(G) $g^{-1}(]-\infty,0[)=]\gamma_-,\gamma_+[$ where $-1<\gamma_-<\gamma_+<1$ and for some

$s<1$ we have $|G(u)|\le s\sqrt{8f(u)}$, $\forall u\in\mathbb{R}$, where $G(u):=\int_0^u g(s)\,ds$.

Lemma 9. *Under the condition (G), there is a constant $k>0$ such that $\forall u\in\mathcal{E}$*

$$\mathcal{F}(u)\ge k\int_{-\infty}^{+\infty}[\frac{u''^2}{2}+f(u)]\,dx.$$

Proof. Take $c\in]k,1[$ and compute

$$\int_{-\infty}^{+\infty}[\frac{1}{2}(u''^2+g(u)u'^2)+f(u)]\,dx\ge$$

$$\int_{-\infty}^{+\infty}[\frac{1}{2}((1-c^2)u''^2+(cu''-\frac{G(u)}{2c})^2)+(f(u)-\frac{G(u)^2}{8c^2})]\,dx$$

where we have performed integration by parts to obtain

$$-\int_{-\infty}^{+\infty}G(u)u''\,dx=\int_{-\infty}^{+\infty}g(u)u'^2\,dx.$$

Hence by our assumption

$$\mathcal{F}(u)\ge\int_{-\infty}^{+\infty}[\frac{(1-c^2)u''^2}{2}+(1-(\frac{k}{c})^2)f(u)]\,dx.$$

Theorem 10. *Let (F1)-(F2)-(F3)-(G) hold. Then equation (2) has a heteroclinic connecting ±1 that minimizes $\mathcal{F}$ on $\mathcal{E}$.*

Proof. The proof follows the same ideas as in theorem 7. The aim is to modify a minimizing sequence (u_n) for $\mathcal{F}$ so that it is possible to extract converging subsequences.

Step 1. (u_n) is bounded in $C^1(\mathbb{R})$. This is a straightforward consequence of Lemmas 9 and 2.

Step 2. The statements of Lemmas 4 and 5 are true for the functional $\mathcal{F}$. While this is clear with respect to lemma 4 we must make some comment on how to replace the tails of a given function $u=u_n$ for a new function related to u while the value of the functional $\mathcal{F}$ decreases. So let us consider for instance the right tail. Given that $|u(x_2)-1|<\epsilon$ and $|u'(x_2)|<\epsilon$ consider the functional

$$\mathcal{F}_2(v)=\int_{x_2}^{+\infty}[\frac{1}{2}(v''^2+g(v)v'^2)+f(v)]\,dx$$

having as domain D_2 the set of functions $v \in C^1[x_2,+\infty[)$ such that $v'' \in L^2(]x_2,+\infty[)$, $v(x_2)=u(x_2)$ and $v'(x_2)=u'(x_2)$. Using integration by parts as in the proof of Lemma 8 we see that, for any function $v \in D_2$

$$\mathcal{F}_2(v) \geq -G(v(x_2))v'(x_2) + k\int_{x_2}^{+\infty} [\frac{v''^2}{2} + f(v)]\, dx. \tag{8}$$

We shall also consider the subset $C_2 \subset D_2$ consisting of those functions v such that $|v(x)-1| \leq a \ \forall x \geq x_2$. Without loss of generality we may assume that a is so small that $g(u) > 0 \ \forall u \in [1-a, 1+a]$.

Claim A. If $v \in D_2 \backslash C_2$ and v' is bounded then $\mathcal{F}_2(v) \geq -G(v(x_2))v'(x_2) + k\frac{a^3}{8N}$. Here N is an upper bound for $\|v'\|_\infty$.

In fact, as we have seen in the proof of Lemma 5, we may choose an interval $[x_3, x_4]$ such that $x_3 \geq x_2$, $v(x_3) = a/2$, $v(x_4) = a$ and $a/2 \leq v(x) \leq a \ \forall x \in [x_3, x_4]$. Hence the result follows as before, by using the above inequality (8).

Claim B. The minimum of $\mathcal{F}_2$ in D_2 exists and, if ϵ is sufficiently small, the minimum is attained at a function $z \in C_2$.

Clearly, $\mathcal{F}_2$ is bounded below in D_2. Estimates analogous to those obtained in Lemma 2 hold. Using inequality (8) it is easily seen that by extracting convergent subsequences from a minimizing sequence the minimum is obtained. Now let $C_2(\delta) = \{v \in D_2 : |v(x)-1| \leq \delta \quad \forall x \geq x_2\}$. Let also $\beta := \sup\{g(u) : u \in [1-a, 1+a]\}$. Given $0 < \delta < a$ there exists $\epsilon > 0$ such that the solution z of the linear problem

$$v^{iv} - \beta v'' + 4\alpha^4(v-1) = 0, \ \ v(x_2) = u(x_2), \ \ v'(x_2) = u'(x_2), \ \ v(+\infty) = 1$$

belongs to $C_2(\delta)$. On the other hand

$$\min_{C_2} \mathcal{F}_\in \leq \min_{C_2(\delta)} \mathcal{F}_2 \leq \min_{v \in C_2(\delta)} \int_{x_2}^{+\infty} [\frac{1}{2}(v''^2 + \beta v'^2) + 2\alpha^4(v-1)^2]\, dx =$$

$$\int_{x_2}^{+\infty} [\frac{1}{2}(z''^2 + \beta z'^2) + 2\alpha^4(z-1)^2]\, dx = O(\epsilon^2) \ \text{ as } \ \epsilon \to 0.$$

Combining this with Claim A, Claim B follows.

Step 3. The minimization procedure completed. As in theorem 7, a minimizing sequence (u_n) for $\mathcal{F}$ is replaced by a new minimizing sequence $(v_n) \in \mathcal{V}(R, \epsilon)$ (with a convenient choice of R and ϵ). The estimates in step 1 allow us to extract a subsequence still denoted (v_n) that converges to some function u weakly (in the same sense as in theorem 7) and C^1-uniformly on compact intervals. To see that v is a minimizer of $\mathcal{F}$ it suffices to note that, since $g(v_n) > 0$ on $[R, +\infty[$, Fatou's lemma is applicable and

$$\int_R^{+\infty} g(u(x))u'(x)^2\, dx \leq \liminf_{n\to\infty} \int_R^{+\infty} g(v_n(x))v_n'(x)^2\, dx.$$

The same is true on $]-\infty, -R]$ and of course this implies that

$$\int_{-\infty}^{+\infty} g(u(x))u'(x)^2\, dx \leq \liminf_{n\to\infty} \int_{-\infty}^{+\infty} g(v_n(x))v_n'(x)^2\, dx.$$

Similar arguments can be used to deal with the double well potential. However, we should rephrase (F2') as

$(F2'')$ There exists m such that

$$f(u) = m\frac{u^2}{2}\ \ \forall x \in]-a,a[\ \text{ and } \ f(u)=0 \ \text{ if and only if } \ u=0 \ \text{ or } \ u=\pm 1.$$

The next two lemmas are similar to a part of the contents of section 4 in [3].

Lemma 11. *Let $|\alpha| < 2\sqrt{\beta}$. Then there exists $\Delta > 0$ such that any nontrivial solution of the differential equation*

$$u^{iv} + \alpha u'' + \beta u = 0$$

changes sign in every interval of length $> \Delta$.

Proof. After rescaling we can write any solution as $u = Ae^t \sin(t+\phi) + Be^{-t}\sin(t+\psi)$ where A, B, ϕ, $\psi \in \mathbb{R}$. We can suppose that $A \neq 0$ and $B \neq 0$. Set $E(t) = |A|e^t$, $F(t) = |B|e^{-t}$ and let $\bar{t}$ be such that $F(\bar{t}) = G(\bar{t})$. Since $\max\{F(t), G(t)\} > \min\{F(t), G(t)\}$ for $t \neq \bar{t}$ and the points where the graphs of $Ae^t\sin(t+\phi)$ and $Be^{-t}\sin(t+\psi)$ touch those of $\pm E$ and $\pm F$ have abcissae that difer from π, we see that it suffices to choose $\Delta > 3\pi$.

Lemma 12. *Let $T > 1$ and α, β be as in the preceeding lemma. Let z be a minimizer of*

$$\mathcal{G}_T(u) := \int_0^T [\frac{1}{2}(u''^2 + \alpha u'^2 + \beta u^2)\, dx$$

in the subspace $Z(T,\delta,\eta)$ of $H^2(0,T)$ consisting of those functions such that $u(0) = 0$, $u(T) = \delta$, $u'(T) = \eta$. Then given $\epsilon > 0$ there exist $\delta_0 > 0$ and $\eta_0 > 0$ such that for all $T \geq 1$, if $|\delta| < \delta_0$ and $|\eta| < \eta_0$ then $|z| \leq \epsilon$ in $[0,T]$.

Proof. First note that indeed the minimum exists, since under our assumptions there are constants $c > 0$, $d > 0$, depending on α and β only, such that we have for all $u \in Z(T,\delta,\eta)$

$$\mathcal{G}_T(u) \geq c\|u\|^2_{H^2(0,T)} - d\delta\eta. \tag{9}$$

It is easy to see that the value of the minimum is $O(\delta^2 + \eta^2)$. Now if there is a constant $k > 0$ such that, for all choices of $|\delta|$ and $|\eta|$ the minimizer z satisfies $\|z\|_{C[0,T]} > k$ then (using lemma 1, for instance) we obtain another constant $k' > 0$ such that $\|z\|_{H^2(0,T)} > k'$. From (9) and what has been said above we obtain a contradiction and the lemma follows.

Theorem 13. *Assume that f is even and satisfies (F1),(F2') and (F4). Assume in addition that g is a C^2 even function and satisfies (G). Then there exists an odd heteroclinic of (2) connecting ± 1.*

Proof. As in the preceeding case we concentrate on the details that demand a different argument with respect to the simpler case where $g \equiv 0$. The main idea is to modify a minimizing sequence for

$$\mathcal{F}_0 = \int_0^{+\infty} [\frac{1}{2}[(u''^2) + g(u)u'^2] + f(u)]\, dx.$$

in $\mathcal{E}_0$ so that its modified elements go from 0 to a neighborhood of 1 in bounded time. To simplify our exposition, in way similar to what we have done about f, assume in addition

$$(G') \qquad \exists g_0 < 0 \text{ such that } g(u) = g_0 \ \ \forall u \in]-a, a[.$$

Note that the compatibility condition (G) entails

$$g_0 < 2\sqrt{m}.$$

So fix $0 < \epsilon < a$ and take one of the elements $u \in \mathcal{E}_0$ of a given minimizing sequence. If t^* is as in the proof of theorem 8, consider the largest value $t^+ < t^*$ such that $u(t^+) = \epsilon$ and $u(t) > \epsilon$ whenever $t \in]t^+, t^*[$; it is clear that $t^* - t^+$ is bounded in terms of an upper bound of $\mathcal{F}_0$ taken over the sequence. Now if u has no zero in $[t^+ - 1, t^+]$ there is $T \in [t^+ - 1, t^+]$ where $0 < u(T) = \delta < \epsilon$ and $0 \le u'(T) = \eta < \epsilon$.

Claim: If ϵ is sufficiently small, the minimum of

$$\mathcal{F}_{0,T} = \int_0^{T} [\frac{1}{2}[(u''^2) + g(u)u'^2] + f(u)]\, dx.$$

in $Z(T, \delta, \eta)$ is attained in some function z with $\|z\|_\infty < a$ and so it is the minimum of $\mathcal{G}_T$.

Proof of the Claim: as in lemma… we have

$$\mathcal{F}_{0,T}(u) \ge G(\delta)\eta + k\mathcal{J}_{0,T}(u), \quad u \in Z(T, \delta, \eta)$$

where we have set $k\mathcal{J}_{0,T} = \int_0^T [\frac{1}{2}[u''^2 + f(u)]\, dx$ and we argue using the last summand as in the proof of Claim B in theorem 10.

Suppose that ϵ has been fixed according to the Claim. We replace the restriction of u to $[0, T]$ with the minimizer z of $\mathcal{F}_{0,T}$, so that with the new function

$$w(x) = \begin{cases} z(x) \text{ if } 0 \le x \le T \\ u(x) \text{ if } x \ge T \end{cases}$$

we clearly have $\mathcal{F}_0(w) \le \mathcal{F}_0(u)$. Lemma 11 implies that $w(t^-) = 0$ for some $t^- \ge T - \Delta$. Integration by parts again yields

$$\int_0^{t^-} [\frac{1}{2}[(w''^2) + g(w)w'^2] + f(w)]\, dx \ge 0$$

so that we may discard the restriction of w to $[0, t^-]$ and the function $v(x) = w(x + t^-)$ has the desired properties: $\mathcal{F}_0(v) \leq \mathcal{F}_0(u)$ and it enters the ϵ-neighborhood of (1) at a time depending only of the upper bound of the sequence and the value of ϵ.

References

1. Jan Bouwe van den Berg, *The phase plane picture for a class of fourth order conservative differential equations,* J. Diff. Equations,**161** (2000), 110-153.
2. G. Gomper and M. Schick, *Phase transitions and critical phenomena,* Academic Press 1994.
3. W. D. Kalies, J. Kwapisz and R. C. A. M. VanderVorst, *Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria,* Comm. Math. Physics **193**, (1998), 337-371.
4. H. Leitão, *Estrutura e Termodinâmica de Misturas Ternárias com Anfifílico,* Ph. D. Thesis, Universidade de Lisboa, 1998.
5. H. Leitão and M.M. Telo da Gama, *Scaling of the interfacial tension of microemulsions: A Landau theory approach*, J. of Chemical Physics **108**, (1998), 4189-4198.
6. W. D. Kalies and R. C. A. M. VanderVorst, *Multitransition homoclinic and heteroclinic solutions of the extended Fisher-Kolmogorov equation*, J. Diff. Equations **131**, (1996), 209-228.
7. L. A. Peletier and W. C. Troy, A topological shooting method and the existence of kinks of the extended Fisher-Kolmogorov equation, Topological Methods in Nonlin. Analysis **6**, (1995), 331-355.
8. D. Smets and J. B. van den Berg, *Homoclinic solutions for Swift-Hohenberg and suspension bridges type equations*, preprint.