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A Partial Generalization of Diliberto's Theorem for Certain DEs of Higher Dimension.

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Abstract. For analysis of bifurcation of planar systems is sometimes used a result first obtained probably by Diliberto. This result is here partially extended to certain class of autonomous ordinary differential equations in \mathbb{R}^3 .

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1 Main results

Andronov and his coworkers were able, in their famous book [1], to derive an effective theorem about continuation of periodic solutions of a two dimensional system. Chicone [2] achieved similar result but with substantially greater elegance using a result first published by Diliberto [3]:

Theorem 1. *Let $F = (p, q)$ be a C^1 2-vector function defined on an open subset of \mathbb{R}^2 . Let $(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))$ be the solution of plane initial problem*

$$\begin{aligned} \dot{x} &= p(x, y) & x(0) &= x_0, \\ \dot{y} &= q(x, y) & y(0) &= y_0. \end{aligned} \tag{1}$$

If $|p(x_0, y_0)| + |q(x_0, y_0)| > 0$, then the principal fundamental matrix $Y(t)$ of (1) at $t = 0$ of the homogeneous variational equation

$$\dot{i} = \frac{\partial p(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))}{\partial x} u + \frac{\partial p(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))}{\partial y} v$$

$$\dot{v} = \frac{\partial q(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))}{\partial x} u + \frac{\partial q(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))}{\partial y} v$$

is such that

$$Y(t)F^\top(x, y) = a(t, x, y)F((\varphi(t, x, y), \psi(t, x, y))) + b(t, x, y)F^\top(\varphi(t, x, y), \psi(t, x, y))$$

and $Y(t)F(x, y) = F((\varphi(t, x, y), \psi(t, x, y)))$, where

$$b(t, x, y) = \frac{p^2(x, y) + q^2(x, y)}{p^2(\varphi(t, x, y), \psi(t, x, y)) + q^2(\varphi(t, x, y), \psi(t, x, y))} \times \\ \times \exp \left\{ \int_0^t \frac{\partial p(\varphi(s, x, y), \psi(s, x, y))}{\partial x} + \frac{\partial q(\varphi(s, x, y), \psi(s, x, y))}{\partial y} ds \right\}. \quad (2)$$

In his book Chicone [2] was able to obtain an interesting geometrical identification for the function $b(t, x, y)$.

Nowadays we are able to extend Diliberto's result on many differential systems in \mathbb{R}^n and the results will be published elsewhere. In this short announcement we shall limit ourselves to a three-dimensional system

$$\dot{x} = f(x), \quad (3)$$

fulfilling the following hypotheses:

H1 the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 function with open domain,

H2 all solutions of (3) are defined on $[0, \infty)$,

H3 the system (3) has a nondegenerate first integral ([4, p.114]) $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g \in C^2$.

If $\varphi(t, x)$ is the solution of the initial problem (3), $\varphi(0, x) = x$, then the first integral g yields a two-dimensional submanifold M generated as the level set of g containing the point x . Clearly the unit surface normal $n(x) := \|\text{grad}g(x)\|^{-1}\text{grad}g(x)$ is well defined and is C^1 on M . Moreover we may suppose that

H4 there are two C^1 functions $a_1(x)$, $a_2(x)$ on M such that $\|a_1(x)\| = \|a_2(x)\| = 1$ and $T_x M := \text{span}\{a_1(x), a_2(x)\}$.

Finally denoting the usual inner product as $\langle \cdot, \cdot \rangle$ and the usual cross-product as $[\cdot, \cdot]$, we may state Diliberto's theorem for three-dimensional systems (3) with a first integral.

Theorem 2. *Let hypotheses H1, H2, H3, H4 be fulfilled and $\varphi(t, x)$ denote the solution of the differential equation (3), $\varphi(0, x) = x$. If $f(x) \neq 0$, then the principal fundamental matrix $Y(t)$ at $t = 0$ of the variational equation $\dot{y} = Df(\varphi(t, x))y$ is such that $Y(t)f(x) = f(\varphi(t, x))$,*

$$Y(t)[n(x)|f(x)] = a(t, x)f(\varphi(t, x)) + b(t, x)[n(\varphi(t, x))|f(\varphi(t, x))],$$

and

$$b(t, x) = \frac{\|f(x)\|^2}{\|f(\varphi(t, x))\|^2} \exp \int_0^t \left\{ \langle a_1 | (Df)a_1 \rangle + \langle a_2 | (Df)a_2 \rangle - \langle a_1 | (Da_1)f \rangle + \langle a_2 | (Da_2)f \rangle \right\} (\varphi(s, x)) ds. \quad (4)$$

As an application let us present the following theorem concerning the Poincaré mapping of the system (3):

Theorem 3. *Let the hypotheses H1, H2, H3 and H4 be fulfilled. Let $x_1 \in M$, $f(x_1) \neq 0$ and $x_1 = \varphi(p, x_1)$, where $0 < p < \infty$ is the first time with this property. Let Σ be a plane containing x_1 and orthogonal to $f(x_1)$. Let $\Psi: U \subset \Sigma \rightarrow \Sigma$ be the Poincaré mapping. If $v \in T_{x_1}\Sigma \cap T_{x_1}M$, then*

$$D\Psi(x_1)v = \frac{\langle v_1 | [n(x_1)|f(x_1)] \rangle}{\|f(x_1)\|^2} b(p, x_1)[n(x_1)|f(x_1)].$$

References

1. A. A. ANDRONOV, E. A. LEONTOVICH, I. I. GORDON AND A. G. MAYER: *Theory of Bifurcation of Dynamical System on the Plane*. John Wiley & Sons Inc., New York–London–Sydney, 1973.
2. C. CHICONE: *Ordinary Differential Equations with applications*. Springer-Verlag, New York, 1999.
3. S. P. DILIBERTO: *On Systems of Ordinary Differential equations*. In *Contributions to the Theory of Nonlinear Oscillations*. Annals of Math. Studies **20**, (1950), pp.1–38. Princeton: Princeton University Press.
4. PH. HARTMAN: *Ordinary Differential Equations*., John Wiley & Sons Inc., New York–London–Sydney, 1964.

