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In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 339--347.

Persistent URL: http://dml.cz/dmlcz/700365

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 339–347

Logistic Equation on Measure Chains

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Abstract. The dynamic equation (14) unifying both the Verhulst differential and the Pielou difference logistic equations is derived. Some application of it is briefly discussed.

MSC 2000. 34A99, 39A12, 92D25

Keywords. Logistic equation, continuous and discrete population growth

1 Introduction

The logistic equation

$$x' = rx\left(1 - \frac{x}{K}\right) \tag{1}$$

introduced by Pierre François Verhulst in 1838 [8] has became a useful tool in modeling of a population growth. Here, x denotes a "size" of population (number of individuals, population density, biomass etc.), r an intrinsic growth rate and K a carrying capacity of environment. The equation serves as a "standard equation" in population dynamics and it uses to be generalized in various directions. Among these generalizations, discrete analogies of equation (1) play an important role — such difference equations represents models of populations with non-overlapping generations. There are several discrete logistic equations. The Euler discretization of equation (1) with the step 1 gives the equation

$$x_{k+1} = \varrho x_k\left(1 - \frac{x_k}{K}\right), \tag{2}$$

which was utilized e.g. by Maynard Smith [5]. This equation has one great disadvantage: if the population size is greater than K, it become negative in the next

* Research supported by grant 201/99/0295 of the Czech Grant Agency

This is the preliminary version of the paper.

time step. This is why, another discrete population models were constructed. The most popular one is the May [4] equation

$$x_{k+1} = x_k \exp\left[\rho\left(1 - \frac{x_k}{K}\right)\right]. \tag{3}$$

A different discrete equation

$$x_{k+1} = \frac{\varrho x_k}{1 + \frac{\varrho - 1}{K} x_k} \tag{4}$$

was proposed by Pielou [6].

Now, the equation arises — which equation among (2), (3) and (4) is a "correct" discrete analogy of equation (1)? By the "correctness", I mean the following property: there exists a general equation such that equation (1) and a difference equation are particular cases of it. Recently, a powerful tool for unification of differential and difference equations was discovered by Stefan Hilger [2]. It is the calculus on a special structure — measure chains or time scales. (The two concepts were used as synonyms in the original Hilger's papers. Now, the words "measure chain" denote an abstract structure while the notion "time scale" stands for a particular subset of reals.)

The unification of discrete and continuous models of population growth has much more than a theoretical (or aesthetic) significance. An evolution of population may be neither completely continuous nor completely discrete. A certain species may evolve in a continuous way during a period of favorable conditions. But such a period may be interrupted by an event or by a season of bad conditions and the size of population "jumps" to a different value after such event or season. As a typical examples, we can consider an insect population taken for a pest in agriculture and hence chemically destroyed in regular or irregular time intervals, or a population of mites which reproduces itself with several generations during spring and summer times and only some fertilized females survive winter season. A unified equation — neither continuous nor discrete but possessing properties of the both cases — should describe such populations, too.

The derivation of unified equation is the main result of the contribution. In the next section, the notion of measure chain is briefly described, the dynamic equations and the exponential function on measure chain are reminded. This section is intended mainly as an "advertisement" of the theory. The calculus on measure chain (time scales) is described in details in the monographs [1] and [3]. The last section contains the announced result.

2 Measure chains

Measure chain is a set $\mathbb{T}$ satisfying the following axioms.

Axiom 1 (Chain) $\mathbb{T}$ is totally ordered set.

This means that there is a relation $\leq$ on $\mathbb{T}$ which is reflexive ($t \leq t$ for all $t \in \mathbb{T}$), antisymmetric ($s \leq t$ and $t \leq s \Rightarrow s = t$), transitive ($r \leq s$ and $s \leq t \Rightarrow r \leq t$) and total ($s \leq t$ or $t \leq s$ for all $t \in \mathbb{T}$).

As usual, $s < t$ means $s \le t$ and $s \ne t$, $s > t$ means $t < s$, and so on. The open, close and half-open intervals of $\mathbb{T}$ are defined by

$$
\begin{aligned}
&[r, s] = \{t \in \mathbb{T} : \, r \le t \le s\}, \\
&]r, s[= \{t \in \mathbb{T} : \, r < t < s\}, \\
&]r, s] = \{t \in \mathbb{T} : \, r < t \le s\}, \quad [r, s[= \ \{t \in \mathbb{T} : \, r \le t < s\}.
\end{aligned}
$$

The *order topology* is generated by the open intervals of $\mathbb{T}$. An open interval of $\mathbb{T}$ containing $t \in \mathbb{T}$ is called *neighborhood of* t.

Axiom 2 (Conditionally complete chain) Each nonempty subset of $\mathbb{T}$, which is bounded above, has a least upper bound (supremum).

Consequently, each nonempty subset of $\mathbb{T}$, which is bounded below, has a greatest lower bound (infimum).

The *forward* and *backward jump operators* are mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : \, s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : \, s < t\}.$$

Via these two operators, points in $\mathbb{T}$ can be classified with respect to their right and left order neighborhood: $t \in \mathbb{T}$ is called *right-dense, right-scattered, left-dense* or *left-scattered*, if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ or $\rho(t) < t$, respectively; $t \in \mathbb{T}$ is called *dense* if it is right-dense or left-dense.

Axiom 3 (Growth calibration) There exists a mapping $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ with the following properties

– (Cocycle property) For all $r, s, t \in \mathbb{T}$ we have $\mu(r, s) + \mu(s, t) = \mu(r, t)$.
– (Strong isotony) For all $r, s \in \mathbb{T}$ we have the implication $r > s \Rightarrow \mu(r, s) > 0$.
– (Continuity) μ is continuous with respect to the product order topology.

The function $d(r, s) := |\mu(r, s)|$ is a metric on $\mathbb{T}$ which generates the order topology. Because of the conditionally completeness with respect to the ordering, $\mathbb{T}$ is also complete with respect to the metric.

The graininess function $\mu^* : \mathbb{T} \to \{x \in \mathbb{R} : \, x \ge 0\}$ is defined as

$$\mu^*(t) = \mu(\sigma(t), t).$$

Any closed subset of $\mathbb{R}$ with the natural ordering and with growth calibration μ defined by $\mu(r, s) = r - s$ is an example for a measure chain; it is called a *time scale.* If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu^*(t) = 0$ for each $t \in \mathbb{T}$; if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$ and $\mu^*(t) = 1$ for each $t \in \mathbb{T}$. It is remarkable that there is no algebraic structure (e.g. group or ring structure) on $\mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous (right-dense-continuous)*, if it is continuous in each right dense instant and has a left-sided limit in each instant of $\mathbb{T}$, which is at the same time right-scattered and left dense.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided $1 + \mu^*(t) f(t) \ne 0$ for all

$$t \in \mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum } m \\ \mathbb{T}, & \text{otherwise} \end{cases}.$$

In the subsequent text, for function $f : \mathbb{T} \to \mathbb{R}$, we will abbreviate $f \circ \sigma$ by f^σ.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *(delta) differentiable at* $t \in \mathbb{T}$, with *(delta) derivative* $f^\Delta(t) \in \mathbb{R}$, if for each $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there is a neighborhood Ω of t such that

$$\left| f^\sigma(t) - f(s) - f^\Delta(t)\mu(\sigma(t), s) \right| \le \varepsilon |\mu(\sigma(t), s)|$$

for all $s \in \Omega$. It follows that

$$f^\sigma(t) = f(t) + \mu^*(t) f^\Delta(t) \tag{5}$$

for each $t \in \mathbb{T}$ where the derivative exists, see e.g. [3, Theorem 1.2.2]. Let $f, g : \mathbb{T} \to \mathbb{R}$ be functions, which are differentiable at $t \in \mathbb{T}$ and let $c \in \mathbb{R}$. Then the following hold

$$(cf(t))^\Delta = cf^\Delta(t), \tag{6}$$

$$(f(t) \pm g(t))^\Delta = f^\Delta(t) \pm g^\Delta(t), \tag{7}$$

$$(f(t)g(t))^\Delta = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t), \tag{8}$$

$$\left(\frac{1}{f(t)}\right)^\Delta = -\frac{f^\Delta(t)}{f^\sigma(t)f(t)} \quad \text{if } f(t) \ne 0 \ne f^\sigma(t). \tag{9}$$

Let $\tau \in \mathbb{T}^\kappa$. If $g : \mathbb{T}^\kappa \to \mathbb{R}$ is an rd-continuous function, then there exists exactly one function $f : \mathbb{T} \to \mathbb{R}$ with the properties:

$$f^\Delta(t) = g(t) \text{ for all } t \in \mathbb{T} \quad \text{and } f(\tau) = 0,$$

see e.g. [3, Theorem 1.4.4]. According to the corresponding notation in real analysis we write

$$\int_\tau^t g(s)\Delta s = f(t).$$

Applying (5), we obtain the useful formula

$$\int_\tau^t g(s)\Delta s = \int_\tau^{\sigma(t)} g(s)\Delta s - \mu^*(t)g(t). \tag{10}$$

The basic concepts concerning measure chains and the two important special cases of it — i.e. the sets of reals $\mathbb{R}$ and integers $\mathbb{Z}$ — are summarized in table 1.

Example 1. Let $\{a_k\}_{k=0}^\infty$ be an increasing sequence of reals and let $\{d_k\}_{k=1}^\infty$ be a sequence of positive reals. Let $\mathbb{T} = \{(k, \tau) \in \mathbb{N}_0 \times \mathbb{R} : a_k \le \tau \le a_{k+1}\}$. Let us define the lexicographic ordering on $\mathbb{T}$, i.e.

$$(k, \xi) \le (l, \eta) \quad \Leftrightarrow \quad k < l \text{ or } (k = l \text{ and } \xi \le \eta)$$

general $\mathbb{T}$	$\mathbb{T}=\mathbb{R}$	$\mathbb{T}=\mathbb{Z}$
forward jump operator $\sigma(t)$	t	$t+1$
backward jump operator $\rho(t)$	t	$t-1$
growth calibration $\mu(r,s)$	$r-s$	$r-s$
graininess function $\mu^*(t)$	0	1
"shift" $f^\sigma(t)$	$f(t)$	$f(t+1)$
rd-continuous f	continuous f	any f
regressive f	any f	$f(t)\neq -1$
derivative $f^\Delta(t)$	$f'(t)$	$\Delta f(t)=f(t+1)-f(t)$
integral $\int\limits_\tau^t f(s)\Delta s$	$\int\limits_\tau^t f(s)\mathrm{d}s$	$\sum\limits_{k=\tau}^{t-1} f(k)$

Table 1. Two most important special cases of measure chains

and a mapping $\mu : \mathbb{T}\times\mathbb{T}\to\mathbb{R}$ as $\mu\left((k,\xi),(l,\eta)\right)=\xi-\eta+\sum\limits_{i=l+1}^{k} d_i$ (we use the convention: $\sum\limits_{i=n}^{m}\alpha_i = -\sum\limits_{i=m+1}^{n-1}\alpha_i$ for $m<n-1$ and $\sum\limits_{i=n}^{n-1}\alpha_i = 0$). Then $\mathbb{T}$ is a measure chain with growth calibration μ. The forward jump operator σ and the graininess function μ^* are given by

$$\sigma(k,\tau)=\begin{cases}(k,\tau), & \text{if } \tau\neq a_{k+1}\\ (k+1,a_{k+1}), & \text{if } \tau=a_{k+1}\end{cases}, \qquad \mu^*(k,t)=\begin{cases}0, & \text{if } \tau\neq a_{k+1}\\ d_{k+1}, & \text{if } \tau=a_{k+1}\end{cases}.$$

This measure chain can underlay a model of a process whose continuous evolution is usually interrupted by an event (impulse, catastrophe etc.) of "size" d_i at time instants $a_i,\ i=1,2,\dots$. In such a case, (k,τ) denotes a time instant with "absolute time distance" τ from beginning a_0 after k occurrences of the interrupting events. (This example was introduced in [7], but it contained an awkward misprint in the graininess function there.)

Let $f:\mathbb{T}\to\mathbb{R}$ be a function. Then

$$f^\Delta(k,\tau)=\begin{cases}\partial f(k,\tau)/\partial\tau, & \text{if } \tau\neq a_{ka+1} \text{ and } \partial f(k,\tau)/\partial\tau \text{ exists}\\ \left[f(k+1,\tau)-f(k,\tau)\right]/d_{k+1}, & \text{if } \tau=a_{k+1},\end{cases}$$

$$\int\limits_{(0,a_0)}^{(k,\tau)} f(l,s)\Delta(l,s)=\sum_{i=0}^{k-1}\int\limits_{a_i}^{a_{i+1}} f(i,\sigma)\mathrm{d}\sigma+\int\limits_{a_k}^{\tau} f(i,\sigma)\mathrm{d}\sigma+\sum_{i=1}^{[\tau]} d_{i+1}f(i,a_{i+1}).$$

A dynamic equation is an equation of the form $x^\Delta=f(t,x)$, where the mapping $f\colon\mathbb{T}\times\mathbb{R}\to\mathbb{R}$. Let $p\colon\mathbb{T}\to\mathbb{R}$. The dynamic equation

$$x^\Delta=p(t)x \tag{11}$$

is called a *(first order) linear (dynamic) equation.* If function p is regressive and rd-continuous, then the initial value problem (11), $x(t_0) = x_0$ admits exactly one solution, see e.g. [1, Theorems 2.33, 2.35].

Let $p : \mathbb{T} \to \mathbb{R}$ be an rd-continuous regressive function. The *(generalized) exponential function* $\mathrm{e}_p(t,\tau)$ is defined to be the unique solution of the initial value problem

$$x^{\Delta} = p(t)x, \quad x(\tau) = 1.$$

Clearly, the derivative of the exponential function is given by

$$\mathrm{e}_p^{\Delta}(t,\tau) = p(t)\mathrm{e}_p(t,\tau). \tag{12}$$

If $\mathbb{T} = \mathbb{R}$, $p \equiv 1$ and $\tau = 0$, then the solution of initial value problem $x' = x$, $x(0) = 1$ is $x(t) = \exp(t)$. This observation justifies the terminology. If $\mathbb{T} = \mathbb{R}$, p is arbitrary continuous function and $\tau = \mathbb{R}$, then

$$\mathrm{e}_p(t,\tau) = \exp \int_{\tau}^{t} p(s)\mathrm{d}s.$$

Example 2. Let $\mathbb{T}$ be the time scale introduced in Example 1 and suppose that the sequence $\{d_k\}_{k=1}^{\infty}$ is bounded above, i.e. $\tilde{d} = \inf\{1/d_k : k = 1, 2, \dots\} > 0$. Further, let $\beta > 0$, $\delta \in]0, \tilde{d}[$ and put

$$r(k,\tau) = \begin{cases} \beta, & \text{if } \tau \neq a_{k+1} \\ -\delta, & \text{if } \tau = a_{k+1} \end{cases}.$$

Then the function r is rd-continuous and regressive. Now, one can easily verify that

$$\mathrm{e}_r\big((k,\tau),(0,a_0)\big) = \Big(\exp(-\beta a_0) - \delta \sum_{i=1}^{k} d_i \exp(-\beta a_i)\Big) \exp(\beta\tau).$$

In particular, if $a_i = i$ for $i \in \mathbb{N}_0$ and $d_i = 1$ for $i \in \mathbb{N}$ then

$$\mathrm{e}_r\big((k,\tau),(0,0)\big) = \Big(1 + \delta \frac{1 - \exp(-\beta k)}{1 - \exp(\beta)}\Big) \exp(\beta\tau).$$

3 The equation

The initial value problem for the nonautonomous logistic ordinary differential equation

$$x' = r(t)x\Big(1 - \frac{x}{K(t)}\Big), \quad x(t_0) = x_0$$

has the solution

$$x(t) \;=\; x_0 \frac{\exp \int_{t_0}^{t} r(\tau)\mathrm{d}\tau}{1 + x_0 \int_{t_0}^{t} \frac{r(\tau)}{K(\tau)} \exp\Big(\int_{t_0}^{\tau} r(s)\mathrm{d}s\Big)\mathrm{d}\tau}.$$

Consequently, a dynamic equation possessing the Verhulst equation (1) as a particular case, should have the solution

$$x(t) = x_0 \frac{\mathrm{e}_r(t,t_0)}{1 + x_0 \int\limits_{t_0}^{t} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0) \Delta\tau}. \tag{13}$$

By (6)–(9) and (12) the delta derivative of the previous equality is

$$x^{\Delta}(t) = x_0 \frac{\mathrm{e}_r^{\Delta}(t,t_0)\Big(1 + x_0 \int\limits_{t_0}^{t} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau\Big) - \mathrm{e}_r(t,t_0) x_0 \frac{r(t)}{K(t)} \mathrm{e}_r(t,t_0)}{\Big(1 + x_0 \int\limits_{t_0}^{t} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau\Big)\Big(1 + x_0 \int\limits_{t_0}^{\sigma(t)} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau\Big)} =$$

$$= x_0 \frac{\mathrm{e}_r(t,t_0)\Big[r(t)\Big(1 + x_0 \int\limits_{t_0}^{t} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau\Big) - x_0 \frac{r(t)}{K(t)} \mathrm{e}_r(t,t_0)\Big]}{\Big(1 + x_0 \int\limits_{t_0}^{t} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau\Big)\Big(1 + x_0 \int\limits_{t_0}^{\sigma(t)} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau\Big)} =$$

$$= x(t) \frac{r(t)\Big(1 + x_0 \int\limits_{t_0}^{t} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau - \frac{x_0}{K(t)} \mathrm{e}_r(t,t_0)\Big)}{\Big(1 + x_0 \int\limits_{t_0}^{\sigma(t)} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau\Big)}$$

Now, formulae (10), (12) and (5) yield

$$x^{\Delta}(t) = r(t)x(t)\left(1 - \frac{x_0 \mu^*(t) \frac{r(t)}{K(t)} \mathrm{e}_r(t,t_0) + \frac{x_0}{K(t)} \mathrm{e}_r(t,t_0)}{1 + x_0 \int\limits_{t_0}^{\sigma(t)} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau}\right) =$$

$$= r(t)x(t)\left(1 - \frac{x_0}{K(t)} \frac{\mu^*(t)\mathrm{e}_r^{\Delta}(t,t_0) + \mathrm{e}_r(t,t_0)}{1 + x_0 \int\limits_{t_0}^{\sigma(t)} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau}\right) =$$

$$= r(t)x(t)\left(1 - \frac{x_0}{K(t)} \frac{\mathrm{e}_r^{\sigma}(t,t_0)}{1 + x_0 \int\limits_{t_0}^{\sigma(t)} \frac{r(\tau)}{K(\tau)} \mathrm{e}_r(\tau,t_0)\Delta\tau}\right) =$$

$$= r(t)x(t)\left(1 - \frac{x^{\sigma}(t)}{K(t)}\right) = r(t)x(t)\left(1 - \frac{x(t) + \mu^*(t)x^{\Delta}(t)}{K(t)}\right) =$$

$$= r(t)x(t)\left(1 - \frac{x(t)}{K(t)}\right) - \mu^*(t)\frac{r(t)x(t)}{K(t)}x^{\Delta}(t).$$

Consequently,

$$x^{\Delta} = \frac{r(t)x\left(1 - \dfrac{1}{K(t)}x\right)}{1 + \mu^*(t)\dfrac{r(t)}{K(t)}x}. \tag{14}$$

Hence, the dynamic equation possessing the solution given by (13) has the form (14).

If $\mathbb{T} = \mathbb{R}$ then $\mu^*(t) = 0$, $x^{\Delta} = x'$ and we get the equation

$$x' = r(t)x\left(1 - \frac{1}{K(t)}x\right).$$

If $\mathbb{T} = \mathbb{Z}$ then $\mu^*(t) = 1$, $x^{\Delta} = x(t+1) - x(t)$. Denoting as usual, $x_k = x(k)$, $r_k = r(k)$, $K_k = K(k)$, we get the equation

$$x_{k+1} = \frac{r_k x_k\left(1 - \dfrac{1}{K_k}x_k\right)}{1 + \dfrac{r_k}{K_k}x_k} + x_k = \frac{(1 + r_k)x_k}{1 + \dfrac{r_k}{K_k}x_k}.$$

The obtained results show that the Verhulst (1) and the Pielou (4) logistic equations are particular cases of dynamic equation (14).

Remark 3. The fact that the equations (1) and (4) are in certain sense the same is not striking at all. The both equations are Riccati ones.

Example 4. Let $\mathbb{T}$ be the time scale introduced in Example 1 with $a_0 = 0$, $a_i = i$, $d_i = 1$ for each $i \in \mathbb{N}$,

$$r(k,\tau) = \begin{cases} \beta, & \tau \neq k+1 \\ -\delta, & \tau = k+1, \end{cases} \quad \text{where} \quad \beta > 0,\ \delta \in [0, 1[.$$

The initial value problem

$$x^{\Delta} = \frac{r(k,\tau)x(1-x)}{1 + \mu^*(k,\tau)r(k,\tau)x}, \quad x(0,0) = x_0 \in]0, 1[$$

can serve to model an evolution of population with intrinsic growth rate β which is regularly exterminated by a chemical compound of efficiency δ. Its solution can be evaluated using the results of Examples 1 and 2.

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