Jiří Vala

Two-scale convergence with respect to measures in continuum mechanics

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 413--435.

Persistent URL: http://dml.cz/dmlcz/700373

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 413–435

Two-scale convergence with respect to measures in continuum mechanics

Jiří Vala *

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering,
Technical University, Žižkova 17, 662 37 Brno, Czech Republic,
Email: Vala.J@fce.vutbr.cz

Abstract. The homogenization of variational formulations of nonlinear systems of partial differential equations with periodically oscillating coefficients, needed in many problems of continuum mechanics with non-negligible microstructural material properties, is studied, using the technique of the two-scale convergence with respect to Radon measures. (Unlike the classical approach, such technique can handle e. g. “domains with holes”, applied in problems of flow of a liquid through a porous medium, without artificial geometrical assumptions.) The overview of basic lemmas (including corresponding proofs) is presented. The existence and convergence analysis for the variational formulation of a model elliptic problem demonstrates how the notion of the two-scale convergence is able to explain and simplify the complicated form of the macroscopic limit equation, thanks to the addition of a new microscopic hidden variable.

MSC 2000. 35J60, 74Q15, 74M25

Keywords. Two-scale convergence, microstructural material properties, elliptic systems of PDEs

1 Remarks to the history of homogenization techniques

In most problems of continuum mechanics at least two length scales can be distinguished – a macroscopic one (usually in meters) and a microscopic one (typically in micrometers), which brings complications to all numerical calculations and simulations. Typically simple algorithms, based on the classical results from textbooks

* Research supported by the Ministry of Education, Youth and Sports of the Czech Republic, Reg. No. CEZ : J22 : 261100007.

This is the final form of the paper.

of functional analysis and numerical methods, then do not reflect physical reality at a satisfactory level; in such sense [6] emphasizes that it is very important to distinguish between the *verification* (whether the ideal mathematical problem is well-defined, solvable, etc.) and the *validation* (whether and how some results similar to the computed ones can be observed both in the laboratories and in the nature) in all engineering applications. In this section we shall mention some practical approaches, their advantages and drawbacks, following their rough classification from [39].

Most commercial software packages and technical standards prefer cheap computations and simple theoretical considerations; only in case when the results are expected to be too far from the realistic ones, some naive “averaging” procedure is applied to material characteristics. Unfortunately, there are many examples of strange and absurd numerical outputs, namely for physical processes in materials consisting of several phases with strongly different mechanical quantities.

Some of these difficulties can be overcome in a relatively simple way, demonstrated for various types of composites e. g. in [40], [35] and [37]: thanks to the assumed periodicity both of a material structure and of external loads, the processes of elastic deformation and high-temperature creep, based on the viscoelastic Maxwell model with one linear elastic component (from the well-known Hooke law) and one non-linear viscous component (e. g. in the power-law form by Norton), coupled with the diffusion and sliding along all phase interfaces, can be analyzed at a microscopic level directly to simulate the effect of thin strengthening (nearly elastic) ceramic plates or fibers, located in a matrix with low creep resistance at elevated temperature and aligned in the proposed direction of uniform tension. Nevertheless, the limits of this access are evident: if some loads are more complicated then the simple extension of results from a micro- to a macroscopic level is not available or requires non-realistic simplifications; attempts to cover both scales using standard FEM, BEM, FVM, RKPM, etc. numerical techniques then lead to unreasonably large, slow and expensive calculations.

A natural way how to handle problems with a periodic microstructure, but without any a priori prescribed macroscopic symmetries, is to improve the “averaging” limit approach using better microstructural information. Such computational homogenization of periodic media has been designed in [5] yet. Consequently a large number of convergence techniques, studying and explaining the limit process from the microscopic (but finite) scale to zero one, has been developed in last two decades: at least the ideas of the asymptotic expansion (cf. [7], [33] and [43]), essentially adapted to the study of periodic problems, and the so-called G-convergence (cf. [34] and [14]), H-convergence (cf. [29]) and Γ-convergence (cf. [15] and [13]) should be mentioned.

These ideas form the theoretical mathematical basis for the study of variational problems with a hidden microstructure, but some their unpleasant common properties cannot be ignored: they are formally complicated and user-unfriendly and the construction of adequate test functions for corresponding integral equations (namely for nonlinear systems) is often tricky, not transparent for physicists. A new approach occurred about ten years ago. All papers and books appreciate the pioneering role of [31], but their definitions, lemmas and theorems are usually taken (and slightly modified or generalized if necessary) from [2]. The main idea is, thanks to the addition of a microscopic hidden variable, to substitute the classical weak and strong convergence in Lebesgue and Sobolev spaces by the so-called two-scale (or multiple-scale) convergence incorporating certain compensated compactness phenomenon due to the particular (not very artificial) choice of test functions. This seems to be equivalent with the original idea of [4], based on certain transform of a spacial variable with respect to a hidden microstructural one, as discussed in [30].

In [10] the two-scale analysis has been applied to the homogenization of several linearized problems, as small-deformation elasticity, heat or wave equation. For linear or quasilinear problems, special families of homogenized FE decompositions supporting the two-scale convergence have been studied in [26] and [27] recently. The mathematical analysis of properties of the two-scale convergence (more detailed than in [2] where some expected result and important proofs are only sketched), extended to parabolic time-dependent problems, is presented in [22]. In many cases of practical interest the two-scale limit passage leads to some effective equations for the original macroscopic problem, but in more complicated nonlinear problems such equations cannot be often written in a simple form, although the two-scale convergence may be guaranteed; this is e. g. the case of the "deck-of-card" model of creep flow applied in [36], whose main idea of "unfractured" (reversible elastic) and "fractured" (irreversible plastic) deformation zones comes from [16].

All above mentioned methods are applicable to domains consisting of several material phases, but without any holes, cracks or perforations. This is evidently not satisfied in problems of flow through porous media where important phenomena occur on the boundaries of pores (both of a macro- and a microscopic size), as demonstrated e. g. in [12] and [38]. In [3] the notion of the two-scale convergence has been generalized from classical non-perforated media from [2] to media with pores described by periodic surfaces; later such access has been used also for selected parabolic time-dependent problems from technical practice (as in [9] or [11] where, in addition, some special nonlinearities are taken into account). These studies introduce various special (and rather strong) assumptions on the shape of a domain under consideration; the proof technique must be then adapted to each case separately. This drawback can be removed by defining the two-scale convergence more carefully with respect to measures (not only in classical Lebesgue spaces as in [2], [22] and their non-substantial modifications). The so-called scale convergence, introduced in [25], can be identified with the rearranged two-scale approach from [2], making use of the properties of Young measures, discussed in [32], with the close relation to the Γ-convergence. In the following sections of this paper we shall deal with a slightly different generalization of this approach, compatible with [8], which is based on its redefinition in Lebesgue spaces (and for gradients in Sobolev spaces) with respect to special periodic Radon measures, applying the tangential calculus, developed in [18]; the weak and strong convergence in such spaces has been characterized in [17]. This approach brings one non-negligible benefit: a corresponding microstructure can include both holes of complicated shapes and parts of lower dimensions without additional geometrical assumptions.

2 Definition and properties of the two-scale convergence

In this section we shall introduce the basic notations, present the definition of the two-scale convergence (with respect to measures) and make the overview of its useful properties, including corresponding proofs, although some of them are slightly modified versions of similar results from [2], [22] or [8]; the main reason for such form of publication is that some well-known lemmas from the classical theory of Lebegue and Sobolev spaces must be checked very carefully in the generalized spaces (and usually some non-standard assumptions are needed).

Let us consider a n-dimensional domain Ω in the Euclidean space $\mathbb{R}^n$ ($n \in \{1,2,3\}$) with a boundary $\partial\Omega$; we shall use Cartesian coordinates in $\mathbb{R}^n$ everywhere. The standard notation of function spaces (where fixed real numbers $p > 1$ and $q = p/(p-1)$ occur) will be applied without comments and explanations (for more information see e. g. [32], p. 35) including such basic facts from functional analysis as the Hölder inequality (cf. [24], p. 65); the notation of spaces with respect to measures is taken from [18], the lower index $_\#$ forces periodicity. Let Y be a unite cube in $\mathbb{R}^n$ with a boundary $\partial\mathrm{Y}$.

Let μ be some positive Y-periodic Radon measure in $\mathbb{R}^n$ and λ a Lebesgue measure in $\mathbb{R}^n$. Let us choose an arbitrary positive ε. Let μ_ε be a measure ("ε-scaling of μ") defined by the formula

$$\int_\Omega \varphi(x)\,\mathrm{d}\mu_\varepsilon(x) = \varepsilon^n \int_\Omega \varphi(x)\,\mathrm{d}\mu(x/\varepsilon) \quad \forall\,\varphi \in C_0(\Omega)$$

such that for some positive constant ν

$$\int_\Omega |\psi(x,x/\varepsilon)|^q\,\mathrm{d}\mu_\varepsilon(x) \le \nu \sup_{y\in\mathrm{Y}} \int_\Omega |\psi(x,y)|^q\,\mathrm{d}\lambda(x) \quad \forall\,\psi \in L^q_\lambda(\Omega, C_\#(\mathrm{Y})) \qquad (1)$$

holds independently of ε. In the following text all underlined symbols should be understood as sequences indexed with respect to selected positive ε or δ decreasing to zero and all overlined symbols as sequences indexed with respect to integer r increasing to ∞. For simplicity let us assume $\mu(\mathrm{Y}) = 1$ and $\mu(\partial\mathrm{Y}) = 0$. This forces e. g. the convergence of $\underline{\mu_\varepsilon}$ to λ in sense

$$\lim_{\varepsilon\to 0} \int_\Omega |v(x)|^p\,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega |v(x)|^p\,\mathrm{d}\lambda(x) \quad \forall\,v \in L^p_\lambda(\Omega)\,; \qquad (2)$$

more information about such convergence in the vague topology of measures (understood as in [23], p. 120) can be found in [8], p. 1200.

Moreover in Lemma 12 one additional assumption on the connectedness of μ (cf. [8], p. 1210) will be needed: let there exist such positive constant c that the implication (the Poincaré-type inequality)

$$\int_\mathrm{Y} \varphi(y)\,\mathrm{d}\mu(y) = 0 \quad \Rightarrow \quad \int_\mathrm{Y} |\varphi(y)|^p\,\mathrm{d}\mu(y) \le c \int_\mathrm{Y} |\nabla_\mu \varphi(y)|^p\,\mathrm{d}\mu(y) \qquad (3)$$

is valid for any $\varphi \in H^{1\,p}_{\mu\#}(\mathrm{Y})$.

Now we are ready to introduce the boundedness, the two-scale convergence and the strong two-scale convergence of sequences from $L^p_{\mu_\varepsilon}(\Omega)$ with respect to a measure μ:

Definition 1 (*boundedness*)**.** Let $\underline{u}_\varepsilon$ be a sequence in $L^p_{\mu_\varepsilon}(\Omega)$. We say that $(\underline{u}_\varepsilon, \mu_\varepsilon)$ is *bounded* iff there exists such positive constant ϖ that

$$\int_\Omega |u_\varepsilon(x)|^p \,\mathrm{d}\mu_\varepsilon(x) \leq \varpi \tag{4}$$

for all elements of $\underline{u}_\varepsilon$ and any positive ε.

Definition 2 (*two-scale convergence*)**.** Let $\underline{u}_\varepsilon$ be a sequence in $L^p_{\mu_\varepsilon}(\Omega)$. We say that it *two-scale converges* to some $u_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ (briefly $\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u_0$) iff

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\,\psi(x, x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega \int_\mathrm{Y} u_0(x,y)\,\psi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) \tag{5}$$

for every test function $\psi \in L^q_\lambda(\Omega, C_\#(\mathrm{Y}))$.

Definition 3 (*strong two-scale convergence*)**.** Let $\underline{u}_\varepsilon$ be a sequence in $L^p_{\mu_\varepsilon}(\Omega)$ that two-scale converges to some $u_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ (in sense of Definition 2). We say that it *strongly two-scale converges* to u_0 (briefly $\underline{u}_\varepsilon \twoheadrightarrow u_0$) iff

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x)|^p \,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega \int_\mathrm{Y} |u_0(x,y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\,. \tag{6}$$

Remark 4. Notice that (6) in Definition 3 can be rewritten in the weaker form

$$\limsup_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x)|^p \,\mathrm{d}\mu_\varepsilon(x) \leq \int_\Omega \int_\mathrm{Y} |u_0(x,y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)$$

because $\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u_0$ implies

$$\liminf_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x)|^p \,\mathrm{d}\mu_\varepsilon(x) \geq \int_\Omega \int_\mathrm{Y} |u_0(x,y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)$$

automatically; for details see [8], p. 1202.

Remark 5. Obviously for a stationary sequence $\underline{u}_\varepsilon$ where $u_\varepsilon(x) := v(x)$ and a test function $\psi(x,y) = |v(x)|^{p/q} \operatorname{sgn} v(x)$ (independent of y) (5) with $u_0(x,y) = v(x)$ degenerates to (2). More generally: observe that if $u_0 \in C(\Omega, C_\#(\mathrm{Y}))$ then $\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u_0$ whenever $u_\varepsilon(x) := u_0(x, x/\varepsilon)$ for all $x \in \Omega$. If moreover $u_0 \in C(\overline{\Omega}, C_\#(\mathrm{Y}))$ then $\underline{u}_\varepsilon \twoheadrightarrow u_0$; for details see [8], p. 1203. The same is also true in case $u_0(x,y) = u_1(x)u_2(y)$ for all $x \in \Omega$ and $y \in \mathrm{Y}$ where $u_1 \in L^p_\lambda(\Omega)$ and $u_2 \in C_\#(\mathrm{Y})$.

Remark 6. It is easy to see that Definitions 1, 2 and 3 can be modified to cover the case that a sequence $\underline{u}_\varepsilon$ belongs to $L^p_{\mu_\varepsilon}(\Omega)^n$ and u_0 to $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y})^n)$. Remarks 4, 5 and 8 and Lemmas 7, 9, 10 and 11 can be then reformulated without any difficulties. Another simple modification of Definitions 1, 2 and 3 exchanges p and q mutually. (The first type of generalization will be needed e. g. in Lemma 12, the second one in Lemma 10.)

In the following text scalar products in $\mathbb{R}^j$ for $j = n$ or $j = n^2$ will be emphasized by $\cdot$ signs (unlike norms in the usual $|.|$ notation). If $\psi \in L^p_{\mu\#}(\mathrm{Y})$ then the index $_\star$ in $\psi_\star$ will emphasize that $\psi_\star$ is considered as a constant (in 2-nd variable) extension of ψ from Ω onto $\Omega \times \mathrm{Y}$. Standard symbols $\rightarrow$ and $\rightharpoonup$ for the strong and weak convergence in various Banach spaces are used, too. In the rest of this section, assuming that $\underline{u}_\varepsilon$ is an arbitrary sequence in $L^p_{\mu_\varepsilon}(\Omega)$ and $\underline{v}_\varepsilon$ (if needed) an arbitrary sequence in $L^q_{\mu_\varepsilon}(\Omega)$, we shall derive the most interesting and useful properties of two-scale convergent sequences:

Lemma 7 (*on test functions*)**.** *$C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$ is dense in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$. Consequently, if $(\underline{u}_\varepsilon, \mu_\varepsilon)$ is bounded and (5) from Definition 2 is true for any $\psi \in C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$ and certain $u_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ then $\underline{u}_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0$.*

Proof. The density of $C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$ in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ follows from [24], p. 73, and [8], p. 1204. We must only prove that if (5) holds for any $\psi \in C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$ then it is true also for arbitrary $\psi \in L^q_\lambda(\Omega, C_\#(\mathrm{Y}))$. Let us consider

$$
\begin{aligned}
&\int_\Omega u_\varepsilon(x)\,\psi(x, x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega \int_\mathrm{Y} u_\varepsilon(x)\,(\psi(x, x/\varepsilon) - \psi^r(x, x/\varepsilon))\,\mathrm{d}\mu_\varepsilon(x)\\
&+ \int_\Omega u_\varepsilon(x)\,\psi^r(x, x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) - \int_\Omega \int_\mathrm{Y} u_0(x,y)\,\psi^r(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\\
&+ \int_\Omega \int_\mathrm{Y} u_0(x,y)\,(\psi^r(x,y) - \psi(x,y))\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\\
&+ \int_\Omega \int_\mathrm{Y} u_0(x,y)\psi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)
\end{aligned}
$$

where $\overline{\psi}^r \subset C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$ and $\overline{\psi}^r \rightarrow \psi$ in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ (using the density of $C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$ in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$). Since

$$
\lim_{r\rightarrow\infty} \sup_{y\in\mathrm{Y}} \int_\Omega |\psi(x,y) - \psi^r(x,y)|^q\,\mathrm{d}\mu_\varepsilon(x) = 0\,,
$$

the first right-hand-side integral vanishes thanks to the assumed boundedness and (1) (with help of the Hölder inequality) for $r \rightarrow \infty$. Similarly the convergence $\overline{\psi}^r \rightarrow \psi$ implies that the fourth integral can be removed, too. But Definition 3 guarantees that the second and third integrals together tend to zero for each integer r if $\varepsilon \rightarrow 0$. Thus, the limit process $r \rightarrow \infty$ and $\varepsilon \rightarrow 0$ yields (5) with arbitrary $\psi \in L^p_\lambda(\Omega, C_\#(\mathrm{Y}))$.

Remark 8. Various other density results can be found in the cited references; e. g. in the proof of Theorem 15 we shall need the density of $L^p_\lambda(\Omega, C_\#(\mathrm{Y}))$ both in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ and in $L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y}))$.

Lemma 9 (*on compactness***).** *If $(\underline{u}_\varepsilon, \mu_\varepsilon)$ is bounded then there exists such $u_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ that, up to a subsequence, $\underline{u}_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0$.*

Proof. By (1) the choice of a measure μ guarantees the estimate

$$\|\psi_\varepsilon\|_{L^q_{\mu_\varepsilon}(\Omega)} \le \nu^{1/q} \|\psi\|_{L^q_\lambda(\Omega, C_\#(\mathrm{Y}))}$$

for each $\psi \in L^q_\lambda(\Omega, C_\#(\mathrm{Y}))$ where $\psi_\varepsilon(x) := \psi(x, x/\varepsilon)$. For arbitrary positive ε let us introduce a linear operator T_ε applied to arguments $\psi \in L^q_\lambda(\Omega, C_\#(\mathrm{Y}))$, using the Riesz representation theorem (cf. [21], p. 33)

$$[T_\varepsilon, \psi] := \int_\Omega u_\varepsilon(x)\psi_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x)\,.$$

Thus, due to the inequality (4) from Definition 1, we can estimate

$$|[T_\varepsilon, \psi]| \le \|u_\varepsilon\|_{L^p_{\mu_\varepsilon}(\Omega)} \|\psi_\varepsilon\|_{L^q_{\mu_\varepsilon}(\Omega)} \le \varpi^{1/p}\nu^{1/q} \|\psi\|_{L^q_\lambda(\Omega, C_\#(\mathrm{Y}))}\,;$$

this guarantees that $\underline{T}_\varepsilon$ is a bounded sequence in the space dual to $L^q_\lambda(\Omega, C_\#(\mathrm{Y}))$ which can be identified with $L^p_\lambda(\Omega, M_\#(\mathrm{Y}))$ where $M_\#(\mathrm{Y})$ is the space of periodic Radon measures on Y (for details see [2], p. 1486, and [32], p. 40). In virtue of the Alaoglu theorem (see [21], p. 45) then a subsequence from $\underline{T}_\varepsilon$ convergent in the weak $*$ topology can be extracted; this implies that such $T_0 \in L^p_\lambda(\Omega, M_\#(\mathrm{Y}))$ exists that, up to a subsequence,

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) &= \lim_{\varepsilon\to 0}\,[T_\varepsilon, \psi] = [T_0, \psi]\\ &= \int_\Omega \int_\mathrm{Y} u_0(x,y)\psi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\end{aligned}$$

where the existence of some $u_0 \in L^p_\lambda(\Omega, C_\#(\mathrm{Y}))$ corresponding to T_0 follows from the Riesz representation theorem again; but this is directly (5) from Definition 2.

Lemma 10 (*on function products***).** *If $\underline{u}_\varepsilon \twoheadrightarrow u_0$ and $\underline{v}_\varepsilon \rightharpoonup\!\!\rightharpoonup v_0$ (cf. Remark 6) for some $u_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ and $v_0 \in L^q_\lambda(\Omega, L^q_{\mu\#}(\mathrm{Y}))$ then*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\,v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega \int_\mathrm{Y} u_0(x,y)\,v_0(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\,.$$

Proof. Making use of the density guaranteed by Lemma 7, we shall apply for each positive ε the decomposition

$$\int_\Omega u_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega (u_\varepsilon(x) - \varphi_{\delta\varepsilon}(x))v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) + \int_\Omega \varphi_{\delta\varepsilon}(x)v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x)$$

where $\underline{\varphi}_\delta \subset C(\Omega, C_\#(\mathrm{Y}))$, $\underline{\varphi}_\delta \to u_0$ in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ and $\varphi_{\delta\varepsilon}(x) := \varphi_\delta(x, x/\varepsilon)$ for all $x \in \Omega$. Definition 2 with respect to Remark 5 gives

$$\begin{aligned}\lim_{\delta,\varepsilon\to 0} \int_\Omega \varphi_{\delta\varepsilon}(x) v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) &= \lim_{\delta\to 0} \int_\Omega \int_\mathrm{Y} \varphi_\delta(x,y) v_0(x,y)\,\mathrm{d}\mu_\varepsilon(y)\,\mathrm{d}\lambda(x)\\ &= \int_\Omega \int_\mathrm{Y} u_0(x,y) v_0(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\,;\end{aligned}$$

thus it is sufficient to prove

$$\lim_{\delta,\varepsilon\to 0} \int_\Omega (u_\varepsilon(x) - \varphi_{\delta\varepsilon}(x)) v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) = 0\,,$$

but this requirement, thanks to the boundedness of $(\underline{v}_\varepsilon, \mu_\varepsilon)$ by Definition 1 (cf. Remark 6), can be reduced (using the Hölder inequality) to the stronger one

$$\lim_{\delta,\varepsilon\to 0} \|u_\varepsilon - \varphi_{\delta\varepsilon}\|_{L^p_{\mu_\varepsilon}(\Omega)} = 0\,. \tag{7}$$

Two Clarkson inequalities

$$\begin{aligned}&\|u_\varepsilon - \varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)} + \|u_\varepsilon + \varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)} \le 2^{p-1}\left(\|u_\varepsilon\|^p_{L^p_{\mu_\varepsilon}(\Omega)} + \|\varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)}\right)\,,\\ &\|u_\varepsilon - \varphi_{\delta\varepsilon}\|^q_{L^p_{\mu_\varepsilon}(\Omega)} + \|u_\varepsilon + \varphi_{\delta\varepsilon}\|^q_{L^p_{\mu_\varepsilon}(\Omega)} \le 2\left(\|u_\varepsilon\|^p_{L^p_{\mu_\varepsilon}(\Omega)} + \|\varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)}\right)^{q/p}\end{aligned}$$

from [1], p. 37, are available. Using the Definition 3 and taking into account Remark 4, we obtain for $p \le 2$ from the first one

$$\begin{aligned}&\lim_{\delta,\varepsilon\to 0} \|u_\varepsilon - \varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)}\\ &\le \lim_{\delta,\varepsilon\to 0}\left(2^{p-1}\left(\|u_\varepsilon\|^p_{L^p_{\mu_\varepsilon}(\Omega)} + \|\varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)}\right) - \|u_\varepsilon + \varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)}\right)\\ &\le \lim_{\delta\to 0}\left(2^{p-1}\left(\|u_0\|^p_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))} + \|\varphi_\delta\|^p_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))}\right) - \|u_0 + \varphi_\delta\|^p_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))}\right)\\ &\le 2^{p-1}.2\,\|u_0\|^p_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))} - \left(2\,\|u_0\|_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))}\right)^p = 0\end{aligned}$$

and for $p \ge 2$ from the second one (respecting that $q/p = q - 1$)

$$\begin{aligned}&\lim_{\delta,\varepsilon\to 0} \|u_\varepsilon - \varphi_{\delta\varepsilon}\|^q_{L^p_{\mu_\varepsilon}(\Omega)}\\ &\le \lim_{\delta,\varepsilon\to 0}\left(2\left(\|u_\varepsilon\|^p_{L^p_{\mu_\varepsilon}(\Omega)} + \|\varphi_{\delta\varepsilon}\|^p_{L^p_{\mu_\varepsilon}(\Omega)}\right)^{q/p} - \|u_\varepsilon + \varphi_{\delta\varepsilon}\|^q_{L^p_{\mu_\varepsilon}(\Omega)}\right)\\ &\le \lim_{\delta\to 0}\left(2\left(\|u_0\|^p_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))} + \|\varphi_\delta\|^p_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))}\right)^{q/p} - \|u_0 + \varphi_\delta\|^q_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))}\right)\\ &\le 2.2^{q/p}\,\|u_0\|^q_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))} - \left(2\,\|u_0\|_{L^p_\lambda(\Omega,L^p_{\mu\#}(\mathrm{Y}))}\right)^q = 0\,.\end{aligned}$$

Both these results together imply (7).

Lemma 11 (*on strong or weak convergence*)**.** *Let there exist such $u \in L^p_\lambda(\Omega)$ that*

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x)-u(x)|^p\,\mathrm{d}\mu_\varepsilon(x)=0\,. \tag{8}$$

Then $\underline{u_\varepsilon} \twoheadrightarrow u_\star$. Conversely: if $\underline{u_\varepsilon} \rightharpoonup\!\!\rightharpoonup u_0$ for some $u_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ then

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\varphi(x)\,\mathrm{d}\mu_\varepsilon(x)=\int_\Omega \widetilde{u}(x)\varphi(x)\,\mathrm{d}\lambda(x)\quad \forall\varphi\in L^q_\lambda(\Omega) \tag{9}$$

where

$$\widetilde{u}(x):=\int_\mathrm{Y} u_0(x,y)\,\mathrm{d}\mu(y)\,.$$

Proof. By (2) u is included in all $L^p_{\mu_\varepsilon}(\Omega)$ with a positive ε, hence (8) is well-defined. To verify the first assertion with $\underline{u_\varepsilon} \rightharpoonup\!\!\rightharpoonup u_\star$ (instead of its stronger version with $\underline{u_\varepsilon} \twoheadrightarrow u_\star$), by Definition 2 and Lemma 7 it is sufficient to prove

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\psi_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x)=\int_\Omega u(x)\int_\mathrm{Y}\psi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)$$

for any $\psi \in C(\Omega, C_\#(\mathrm{Y}))$ and $\psi_\varepsilon(x) := \psi(x, x/\varepsilon)$. But Remark 5 shows that $\underline{\psi_\varepsilon} \twoheadrightarrow \psi$ (cf. also Remark 6); thus the preceding equation can be rewritten as

$$\lim_{\varepsilon\to 0}\int_\Omega (u_\varepsilon(x)-u(x))\psi_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x)=0$$

and the Hölder inequality gives the expected result. Moreover (using the estimate in the norm of $L^p_{\mu_\varepsilon}(\Omega)$)

$$\lim_{\varepsilon\to 0}\left(\int_\Omega |u_\varepsilon(x)|^p\,\mathrm{d}\mu_\varepsilon(x)\right)^{1/p}\le \lim_{\varepsilon\to 0}\left(\int_\Omega |u_\varepsilon(x)-u(x)|^p\,\mathrm{d}\mu_\varepsilon(x)\right)^{1/p} + \lim_{\varepsilon\to 0}\left(\int_\Omega |u(x)|^p\,\mathrm{d}\mu_\varepsilon(x)\right)^{1/p}.$$

But the first right-hand-side additive term is zero by (8) and $\mathrm{d}\mu_\varepsilon$ in the second one can be replaced by $\mathrm{d}\lambda$ using (2); this with respect to Definition 3 (with $u_0(x,y)$ substituted by $u(x)$ only) completes the proof of the first assertion. The second assertion is a simple consequence of Definition 2 where $\varphi \in L^q_\lambda(\Omega)$ is taken instead of $\psi \in L^q_\lambda(\Omega, C_\#(\mathrm{Y}))$ (being constant in the second variable).

In [2], pp. 1485 and 1488, only the special choice $\underline{\mu_\varepsilon} = \mu = \lambda$ was studied; for this case Lemma 11 guarantees that

$$\underline{u_\varepsilon}\to u \quad\Rightarrow\quad \underline{u_\varepsilon}\twoheadrightarrow u_\star \qquad\text{and}\qquad \underline{u_\varepsilon}\rightharpoonup\!\!\rightharpoonup u_0 \quad\Rightarrow\quad \underline{u_\varepsilon}\rightharpoonup \widetilde{u}$$

in classical sense of strong and weak convergence ($\to$ and $\rightharpoonup$ act in $L^p_\lambda(\Omega)$). In such case a lot of results from the standard theory of Sobolev spaces and functional analysis is available (e. g. Lemma 9 can be understood as a stronger version of the well-known Eberlein - Shmul'yan theorem – cf. [41], p. 201, and [20], p. 197), but this is not true in more complicated cases where namely an appropriate definition of gradients may be far from trivial. We shall introduce the gradients in the same way as in [17], p. 4: for any $\varphi \in H^{1p}_{\mu\#}(\mathrm{Y})$

$$\nabla_\mu \varphi(y) := P_\mu(y)\nabla\varphi(y) \quad \forall\, y \in \mathrm{Y}$$

where $P_\mu \in L^p_{\mu\#}(\mathrm{Y})^{n.n}$ denotes the operator of orthogonal projection onto the local tangent space of μ (defined in [17], p. 3, exactly). Moreover, following [8], p. 1206, we can make use of the operator div_μ coming from the relation of the Green-Ostrogradskiĭ type

$$\int_\mathrm{Y} v\, \mathrm{div}_\mu \Phi \,\mathrm{d}\mu = -\int_\mathrm{Y} \Phi\cdot\nabla_\mu v\,\mathrm{d}\mu \quad \forall\, v \in C^\infty_\#(\mathrm{Y}) \quad \forall\, \Phi \in X^q_{\mu\#}(\mathrm{Y})$$

where the class $X^q_{\mu\#}(\mathrm{Y})$ (of vector fields "tangent to μ") includes all elements of $L^q_{\mu\#}(\mathrm{Y})^n$ whose divergences (in distributional sense) belong to $L^q_{\mu\#}(\mathrm{Y})$; $X^p_{\mu\#}(\mathrm{Y})$ is defined similarly. The space $H^{1p}_{\mu\#}(\mathrm{Y})$ is the completion of $C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$ in the norm

$$\|\varphi\|_{H^{1p}_{\mu\#}(\mathrm{Y})} := \|\varphi\|_{L^p_{\mu\#}(\mathrm{Y})} + \|\nabla_\mu\varphi\|_{L^p_{\mu\#}(\mathrm{Y})^n}$$

for any $\varphi \in C^\infty_0(\Omega, C^\infty_\#(\mathrm{Y}))$; then the space $L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y}))$ can be introduced, too. We intend to apply the projector P_μ especially in another context, taking a macrostructural variable $x \in \Omega$ (not only $y \in \mathrm{Y}$) into consideration: instead of P_μ we shall have P_{μ_ε} with a positive ε such that (similarly to Remark 5, cf. also Remark 6, for details see [8], p. 1203) $\underline{P}_{\mu_\varepsilon} \twoheadrightarrow P_\mu$ and

$$\nabla_{\mu_\varepsilon}\varphi(x) := P_{\mu_\varepsilon}(x)\nabla\varphi(x) \quad \forall\, x \in \Omega$$

for any $\varphi \in H^{1p}_{\mu_\varepsilon}(\Omega)$; moreover if $\varphi_1 \in L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y})^n)$ and $\underline{\varphi}_\varepsilon$ is a sequence of elements from $H^{1p}_{\mu_\varepsilon}(\Omega)$ then

$$\nabla\underline{\varphi}_\varepsilon \rightharpoonup\!\!\rightharpoonup \varphi_1 \quad \Rightarrow \quad \underline{P}_{\mu_\varepsilon}\nabla\underline{\varphi}_\varepsilon \rightharpoonup\!\!\rightharpoonup P_\mu\varphi_1 \tag{10}$$

(this can be verified using Lemma 10). If $\psi \in L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y}))$ then the double dot in $\nabla_{\ddot\mu}\psi$ will indicate that the operator ∇_μ is applied to the second (microstructural) variable only. (In the proof of Theorem 15 also the single or double dot as an index of ∇ will demonstrate that this operator is related to the first or second variable only.) The following important generalization of Lemma 9 studies the behaviour of gradients of some sequences from the point of view of the two-scale convergence:

Lemma 12 (*on gradients*)**.** *Let there exist such positive constant c that (3) is valid for any $\varphi \in H^{1p}_{\mu\#}(\mathrm{Y})$. If $(\underline{u}_\varepsilon, \mu_\varepsilon)$ and $(\nabla_{\mu_\varepsilon}\underline{u}_\varepsilon, \mu_\varepsilon)$ are bounded then some $u \in H^{1p}_\lambda(\Omega)$ and $u_1 \in L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y}))$ exist such that, up to a subsequence, $\underline{u}_\varepsilon \rightharpoonup\!\!\rightharpoonup u_\star$ and $\nabla_{\mu_\varepsilon}\underline{u}_\varepsilon \rightharpoonup\!\!\rightharpoonup \nabla u_\star + \nabla_{\ddot\mu} u_1$ (in sense of Remark 6).*

Proof. By Lemma 9 and Definition 2 (cf. also Remark 6) such $u_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y}))$ and $u_1 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y})^n)$ exist and from $\underline{u}_\varepsilon$ such subsequence can be extracted that $\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u_0$ and $\nabla\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u_1$; in other words: (5) must be true for all $\psi \in L^p_\lambda(\Omega, C_\#(\mathrm{Y}))$ and

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla u_\varepsilon \cdot \Psi(x, x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega\int_\mathrm{Y} u_1(x,y)\cdot\Psi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) \tag{11}$$

for all $\Psi \in L^p_\lambda(\Omega, C_\#(\mathrm{Y})^n)$. Especially for any $x\in\Omega$ and $y\in\mathrm{Y}$ let us choose $\Psi(x,y) = \varphi(x)\Phi(y)$ where $\varphi\in C_0^\infty(\Omega)$ and $\Phi \in X^q_{\mu\#}(\mathrm{Y})$. Following [8], p. 1212, for every positive ε we are able to integrate by parts

$$\begin{aligned}&\varepsilon\int_\Omega \nabla u_\varepsilon\varphi(x)\cdot\Phi(x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) \\ &= -\varepsilon\int_\Omega u_\varepsilon(x)\nabla\varphi(x)\cdot\Phi(x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) - \int_\Omega u_\varepsilon(x)\varphi(x)\,\mathrm{div}_\mu\Phi(x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x)\end{aligned} \tag{12}$$

and thanks to the boundedness of $(\underline{u}_\varepsilon, \mu_\varepsilon)$ and $(\nabla_{\mu_\varepsilon}\underline{u}_\varepsilon, \mu_\varepsilon)$ to obtain

$$\begin{aligned}0 &= \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\varphi(x)\,\mathrm{div}_\mu\Phi(x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) \\ &= \int_\Omega \varphi(x)\int_\mathrm{Y} u_0(x,y)\,\mathrm{div}_\mu\Phi(y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\,.\end{aligned} \tag{13}$$

The existence of a positive constant c from (3) independent of $\varphi\in H^{1p}_{\mu\#}(\mathrm{Y})$ forces that

$$\int_\mathrm{Y}\Phi(y)\cdot\Psi(y)\,\mathrm{d}\mu(y) = 0 \quad \forall\Phi\in W\ \forall\Psi\in W_\perp \tag{14}$$

with

$$\begin{aligned}W &:= \{\Phi\in X^q_{\mu\#}(\mathrm{Y}) :\ \mathrm{div}_\mu\Phi = 0\}\,,\\ W_\perp &:= \{\Psi\in X^p_{\mu\#}(\mathrm{Y}) :\ P_\mu\Psi = \nabla_\mu v \quad \text{for some } v\in H^{1p}_{\mu\#}(\mathrm{Y})\}\end{aligned}$$

and

$$\int_\mathrm{Y} v(y)\,w(y)\,\mathrm{d}\mu(y) = 0 \quad \forall v\in V\ \forall w\in V_\perp \tag{15}$$

with

$$\begin{aligned}V &:= \{v\in L^p_{\mu\#}(\mathrm{Y}) :\ v = \mathrm{div}_\mu\Phi \ \text{ for some } \Phi\in X^p_{\mu\#}(\mathrm{Y})\}\,,\\ V_\perp &:= \{w\in L^q_{\mu\#}(\mathrm{Y}) :\ w = a \ \text{ for some } a\in\mathbb{R}\}\,;\end{aligned}$$

the extensive verification of these seemingly simple facts, generalizing the classical result (“orthogonals of divergence-free functions are exactly the gradients” if $\mu_\varepsilon = \mu = \lambda$) from [2], p. 1492, with $q = 2$ (using the Fourier analysis), and from [22], p. 329, for a general $q > 1$ (based on the absolute continuity on line segments in standard Sobolev spaces by [44], p. 44), to periodic Sobolev spaces with general measures, can be found in [8], p. 1210 (in [42] the assumption $q = 2$ cannot be removed easily). Thus for a fixed $x \in \Omega$ we know from (13) and (15) that $u_0(x,y)$ is a constant on Y which gives a chance to set $u(x) := u_0(x,y)$ independently of $y \in$ Y. In particular let us consider $\Phi \in W$ only. Then (12) (divided by ε) degenerates to

$$\int_\Omega \nabla u_\varepsilon(x)\varphi(x)\cdot\Phi(x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x) = -\int_\Omega u_\varepsilon(x)\nabla\varphi(x)\cdot\Phi(x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x)\,,$$

the limit passage (11) and (5) gives

$$\begin{aligned}\int_\Omega \varphi(x)\int_{\mathrm{Y}} u_1(x,y)\cdot\Phi(y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) &= -\int_\Omega u(x)\nabla\varphi(x)\cdot\int_{\mathrm{Y}}\Phi(y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\\ &= \int_\Omega \varphi(x)\nabla u(x)\cdot\int_{\mathrm{Y}}\Phi(y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\end{aligned}$$

and thanks to the density of $C_0^\infty(\Omega)$ in $L^p_\lambda(\Omega)$ (the choice of φ in $C_0^\infty(\Omega)$ was arbitrary)

$$\int_{\mathrm{Y}} (u_1(x,y) - \nabla u(x))\cdot\Phi(y)\,\mathrm{d}\mu(y) = 0 \quad \text{for } \lambda\text{-a.e. } x\in\Omega\,.$$

¿From (14) we can deduce

$$P_\mu(y)u_1(x,y) - \nabla u(x) = \nabla_{\ddot\mu}u_1(x,y) \quad \text{for } \lambda\text{-a.e. } x\in\Omega \text{ and for } \mu\text{-a.e. } y\in\mathrm{Y}$$

with certain $u_1 \in L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y})^n)$. Let us remember a sequence $\underline{P}_{\mu_\varepsilon}$ from (10). Since $\nabla\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u_1$ (from (11)), $\nabla_{\mu_\varepsilon}\underline{u}_\varepsilon = P_{\mu_\varepsilon}\nabla\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup P_\mu u_1 = \nabla u_\star + \nabla_{\ddot\mu}u_1$.

3 Analysis of a model elliptic problem

To make the notation as clear as possible, we shall introduce some special function classes. Let $\mathrm{Car}^p(\Omega,\mathrm{Y},\mathbb{R}^m)^k$ be a class of functions $b:\Omega\times\mathrm{Y}\times\mathbb{R}^m\mapsto\mathbb{R}^k$ (m and k are positive integers) with the following properties:

(a) b is Y-periodic. Moreover for each $w\in\mathbb{R}^m$ (later $m=n$ or $m=n(1+n)$ together with $k=n$ or $k=n^2$ will be used), for λ-a.e. $x\in\Omega$ and for μ-a.e. $y\in$ Y the growth condition $|b(x,y,w)|\le\beta_b(x,y)+\gamma_b|w|^{p-1}$ is satisfied with some positive γ_b and $\beta_b\in L^q_\lambda(\Omega,C_\#(\mathrm{Y}))$ (by (2) also $\beta_b\in L^q_{\mu_\varepsilon}(\Omega,C_\#(\mathrm{Y}))$ for any positive ε).

(b) $b_\varphi(x,y) := b(x,y,\varphi(x,y))$ applied to all $x\in\Omega$ and $y\in$ Y defines for any $\varphi\in L^p_\lambda(\Omega,C_\#(\mathrm{Y})^m)$ a continuous mapping $b_\varphi : L^p_\lambda(\Omega,C_\#(\mathrm{Y})^m)\mapsto L^q_\lambda(\Omega,C_\#(\mathrm{Y})^k)$.

(In many cases the Nemytskiĭ mappings from [32], p. 36, are useful to verify these properties.) A class $\mathrm{Car}^p(\partial\Omega,\mathbb{R}^n)^n$ can be defined similarly with small changes: in particular $m=k=n$, Ω is replaced by $\partial\Omega$, the modified growth condition from

(a) is valid with some positive γ_g and $\beta_g \in L^p_\sigma(\partial\Omega)$, σ is the Hausdorff measure on $\partial\Omega$, in (a) and (b) the second variable is missing, the requirement on periodicity in (a) disappears.

Now we are ready to formulate our model variational problem:

Problem 13. Find $u \in U_\varepsilon$ such that

$$\int_\Omega a(x, x/\varepsilon, u(x), \nabla_{\mu_\varepsilon} u(x)) \cdot \nabla_{\mu_\varepsilon} v(x) \, \mathrm{d}\mu_\varepsilon(x) \tag{16}$$
$$+ \int_\Omega f(x, x/\varepsilon, u(x)) \cdot v(x) \, \mathrm{d}\mu_\varepsilon(x) + \int_{\partial\Omega} g(x, u(x)) \cdot v(x) \, \mathrm{d}\sigma(x) = 0$$

for all v from certain subspace U_ε of $H^{1p}_{\mu_\varepsilon}(\Omega)^n$ and their suitable extensions to $L^p_\sigma(\partial\Omega)^n$.

To be able to discuss the solvability of this problem, let us suppose:

(c) Some prescribed boundary conditions on $\partial\Omega$ (of the Dirichlet type) force the equivalence of norms

$$\|v\|_{H^{1p}_{\mu_\varepsilon}(\Omega)^n}, \quad \|\nabla_{\mu_\varepsilon} v\|_{L^p_{\mu_\varepsilon}(\Omega)^{n.n}}, \quad \|\nabla_{\mu_\varepsilon} v\|_{L^p_{\mu_\varepsilon}(\Omega)^{n.n}} + \|v\|_{L^p_\sigma(\partial\Omega)^n}$$

for any $v \in U_\varepsilon$ (which can be identified with the Friedrichs-type inequality).

(d) The functions a, f and g (in classical mechanics: stress tensors determined by strain tensors and material characteristics from the constitutive law, volume loads and surface loads) belong to the following classes:

$$a \in \mathrm{Car}^p(\Omega, \mathrm{Y}, \mathbb{R}^{n(1+n)})^{n.n}, \quad f \in \mathrm{Car}^p(\Omega, \mathrm{Y}, \mathbb{R}^n)^n, \quad g \in \mathrm{Car}^p(\partial\Omega, \mathbb{R}^n)^n.$$

The estimates

$$a(x,y,z,\theta) \cdot \theta \geq \kappa|\theta|^p - \vartheta_a(x,y)(|z|^{p-s} + |\theta|^{p-s}) - \zeta_a(x,y),$$
$$f(x,y,z) \cdot z \geq -\vartheta_f(x,y)|z|^{p-s} - \zeta_f(x,y),$$
$$g(\widetilde{x}, z) \cdot z \geq -\vartheta_g(\widetilde{x})|z|^{p-s} - \zeta_g(\widetilde{x})$$

are true for λ-a.e. $x \in \Omega$, for μ-a.e. $y \in \mathrm{Y}$, for σ-a.e. $\widetilde{x} \in \partial\Omega$ and each $z \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^{n.n}$ ($w = (z,\theta)$ from (a) is considered) with some positive constant κ, real s satisfying the inequality $1 < s < p$, ϑ_a and ϑ_f from $L^{p/s}_\lambda(\Omega, C_\#(\mathrm{Y}))$, ϑ_g from $L^{p/s}_\sigma(\partial\Omega)$, ζ_a and ζ_f from $L_\lambda(\Omega, C_\#(\mathrm{Y}))$, ζ_g from $L_\sigma(\partial\Omega)$ (by (2) also ϑ_a and ϑ_f belong to $L^{p/s}_{\mu_\varepsilon}(\Omega, C_\#(\mathrm{Y}))$, ζ_a and ζ_f to $L_{\mu_\varepsilon}(\Omega, C_\#(\mathrm{Y}))$ for any positive ε). Moreover

$$(a(x,y,z,\theta) - a(x,y,z,\widetilde{\theta})) \cdot (\theta - \widetilde{\theta}) > 0 \tag{17}$$

independently of the choice of $\widetilde{\theta} \in \mathbb{R}^{n.n}$ other than θ (i.e. a is strictly monotone).

In the rest of this section (and of the whole paper) we shall derive two results: the first one that Problem 13 with a finite positive ε has already a solution (Theorem 14) and the second one that under certain physically realistic conditions the limit passage $\varepsilon \to 0$ is possible (Theorem 15). These results generalize those from [2], p. 1503, naturally in several directions (probably the most visible is that the assumption "on non-perforation" $\underline{\mu}_\varepsilon = \mu = \lambda$ was removed); the exact formulations and proofs follow:

Theorem 14 (*existence result with a finite* ε). *Let (a), (b), (c) and (d) be satisfied. Then for any positive* ε *Problem 13 has at least one solution.*

Proof. Let $\mathcal{A}_\varepsilon$ be a mapping of U_ε into its dual space by the definition

$$\begin{aligned}\langle \mathcal{A}_\varepsilon u, v\rangle = &\int_\Omega a(x, x/\varepsilon, u(x), \nabla_{\mu_\varepsilon} u(x)) \cdot \nabla_{\mu_\varepsilon} v(x)\, \mathrm{d}\mu_\varepsilon(x) \\ &+ \int_\Omega f(x, x/\varepsilon, u(x)) \cdot v(x)\, \mathrm{d}\mu_\varepsilon(x) + \int_{\partial\Omega} g(x, u(x)) \cdot v(x)\, \mathrm{d}\sigma(x)\end{aligned} \tag{18}$$

for every $u, v \in U_\varepsilon$. By [19], p. 279, if $\mathcal{A}_\varepsilon$ is ⓐ coercive, ⓑ demicontinuous, ⓒ bounded and ⓓ the estimate

$$\limsup_{r\to\infty} \langle \mathcal{A}_\varepsilon v^r - \mathcal{A}_\varepsilon v, v^r - v\rangle \le 0 \tag{19}$$

together with $\overline{v}^r \rightharpoonup v$ forces $\overline{v}^r \to v$ for any sequence $\overline{v}^r \subset U_\varepsilon$ and for a corresponding $v \in U_\varepsilon$ then $\mathcal{A}_\varepsilon$ is also surjective; this implies that the integral equation

$$\langle \mathcal{A}_\varepsilon u, v\rangle = 0 \quad \forall\, v \in U_\varepsilon$$

must have a solution $u \in U_\varepsilon$. Thus the proof of the existence of at least one solution of Problem 13 can be reduced to four steps consisting of the verification of ⓐ, ⓑ, ⓒ and ⓓ:

ⓐ Following [24], p. 65, we shall use the inequality for any positive τ, η and ω

$$\frac{\eta}{\tau^{p-s}}(\tau\omega)^{p-s} \le \frac{s}{p}\left(\frac{\eta}{\tau^{p-s}}\right)^{p/s} + \frac{p-s}{p}(\tau\omega)^p = \frac{s}{p}\tau^{-p(p-s)/s}\eta^{p/s} + \frac{p-s}{p}\tau^p\omega^p$$

valid for any positive τ, η and ω; in the subsequent estimates several times special η and ω will be applied. Let us consider an arbitrary $v \in U_\varepsilon$. ¿From

(d) we receive

$$\int_\Omega a(x,x/\varepsilon,v(x),\nabla_{\mu_\varepsilon}v(x))\cdot\nabla_{\mu_\varepsilon}v(x)\,\mathrm{d}\mu_\varepsilon(x)\ge\kappa\|\nabla_{\mu_\varepsilon}v\|^p_{L^p_{\mu_\varepsilon}(\Omega)^{n\cdot n}}$$
$$-\frac{s}{p}\tau^{-p(p-s)/s}\|\vartheta_a\|^{p/s}_{L^{p/s}_{\mu_\varepsilon}(\Omega,C_\#(\mathrm{Y}))}-\frac{p-s}{p}\tau^p\|v\|^p_{U_\varepsilon}-\|\zeta_a\|_{L_{\mu_\varepsilon}(\Omega,C_\#(\mathrm{Y}))}\,,$$
$$\int_\Omega f(x,x/\varepsilon,v(x))\cdot v(x)\,\mathrm{d}\mu_\varepsilon(x)$$
$$\ge-\frac{s}{p}\tau^{-p(p-s)/s}\|\vartheta_f\|^{p/s}_{L^{p/s}_{\mu_\varepsilon}(\Omega,C_\#(\mathrm{Y}))}-\frac{p-s}{p}\tau^p\|v\|^p_{L^p_{\mu_\varepsilon}(\Omega)}-\|\zeta_f\|_{L_{\mu_\varepsilon}(\Omega,C_\#(\mathrm{Y}))}\,,$$
$$\int_{\partial\Omega} g(x,v(x))\cdot v(x)\,\mathrm{d}\sigma(x)$$
$$\ge-\frac{s}{p}\tau^{-p(p-s)/s}\|\vartheta_g\|^{p/s}_{L^{p/s}_\sigma(\partial\Omega)}-\frac{p-s}{p}\tau^p\|v\|^p_{L^p_\sigma(\partial\Omega)}-\|\zeta_g\|_{L_\sigma(\partial\Omega)}$$

and consequently

$$\langle\mathcal{A}_\varepsilon v,v\rangle\ge\kappa\|\nabla_{\mu_\varepsilon}v\|^p_{L^p_{\mu_\varepsilon}(\Omega)^{n\cdot n}}$$
$$-\frac{p-s}{p}\tau^p\left(\|v\|^p_{U_\varepsilon}+\|v\|^p_{L^p_{\mu_\varepsilon}(\Omega)^n}+\|v\|_{L_\sigma(\partial\Omega)^n}\right)-S_\tau$$

with certain real constant S_τ independent of v. Especially for τ small enough

$$\langle\mathcal{A}_\varepsilon v,v\rangle\ge\kappa_1\|v\|^p_{U_\varepsilon}-\kappa_2 \tag{20}$$

holds by (c) (due to the equivalence of norms) with some positive constants κ_1 and κ_2 independent of v; this implies the coerciveness of $\mathcal{A}_\varepsilon$ (cf. [19], p. 266) evidently.

ⓑ In U_ε let us choose arbitrary u and v and any sequence $\overline{u}^r\to u$. We have

$$\langle\mathcal{A}_\varepsilon u^r-\mathcal{A}_\varepsilon u,v\rangle$$
$$=\int_\Omega\left(a(x,x/\varepsilon,u^r(x),\nabla_{\mu_\varepsilon}u^r(x))-a(x,x/\varepsilon,u(x),\nabla_{\mu_\varepsilon}u(x))\right)$$
$$\cdot\nabla_{\mu_\varepsilon}v(x)\,\mathrm{d}\mu_\varepsilon(x)$$
$$+\int_\Omega\left(f(x,x/\varepsilon,u^r(x))-f(x,x/\varepsilon,u(x))\right)\cdot v(x)\,\mathrm{d}\mu_\varepsilon(x)$$
$$+\int_{\partial\Omega}\left(g(x,u^r(x))-g(x,u(x))\right)\cdot v(x)\,\mathrm{d}\sigma(x)\,.$$

Thus (b) (with help of the Hölder inequality) forces the demicontinuity of $\mathcal{A}_\varepsilon$ (cf. [19], p. 270).

ⓒ Let us consider u and v as in ⓑ again. ¿From (a) with a (where w has to understood as in (d)), f and g substituted to b we obtain

$$|a(x,x/\varepsilon,u(x),\nabla_{\mu_\varepsilon}u(x))|\le\beta_a(x,x/\varepsilon)+\gamma_a\left(|u(x)|^{p-1}+|\nabla_{\mu_\varepsilon}u(x)|^{p-1}\right)\,,$$
$$|f(x,x/\varepsilon,u(x))|\le\beta_f(x,x/\varepsilon)+\gamma_f|u(x)|^{p-1}\,,$$
$$|g(\widetilde{x},u(\widetilde{x}))|\le\beta_g(\widetilde{x})+\gamma_g|u(\widetilde{x})|^{p-1}$$

for λ-a.e. $x \in \Omega$ and for σ-a.e. $\widetilde{x} \in \partial\Omega$ which directly from the definition (18) by the Hölder inequality yields

$$\begin{aligned}\langle \mathcal{A}_\varepsilon u, v\rangle \le & \left(\|\beta_a\|_{L_{\mu_\varepsilon}(\Omega,C_\#(\mathrm{Y}))} + \gamma_a \|u\|^{p-1}_{U_\varepsilon}\right)\|v\|_{U_\varepsilon} \\ & + \left(\|\beta_f\|_{L_{\mu_\varepsilon}(\Omega,C_\#(\mathrm{Y}))} + \gamma_f \|u\|^{p-1}_{L^p_{\mu_\varepsilon}(\Omega)^n}\right)\|v\|_{L^p_{\mu_\varepsilon}(\Omega)^n} \\ & + \left(\|\beta_g\|_{L_\sigma(\partial\Omega)} + \gamma_g \|u\|^{p-1}_{L^p_\sigma(\partial\Omega)}\right)\|v\|_{L^p_\sigma(\partial\Omega)^n}\,.\end{aligned}$$

Making use of (c) we can thus see that independently of u and v such positive β and γ exist that

$$\langle \mathcal{A}_\varepsilon u, v\rangle \le \left(\beta + \gamma\|u\|^{p-1}_{U\varepsilon}\right)\|v\|_{U\varepsilon}\,.$$

Let us introduce the unit ball $\mathcal{B}_\varepsilon := \{v \in U_\varepsilon : \|v\|_{U_\varepsilon} \le 1\}$. The norm of $\mathcal{A}_\varepsilon u$ in the space dual to U_ε then is

$$\sup_{v\in\mathcal{B}_\varepsilon} \langle \mathcal{A}_\varepsilon u, v\rangle \le \beta + \gamma\|u\|^{p-1}_{U_\varepsilon}$$

which guarantees the boundedness of $\mathcal{A}_\varepsilon$ (cf. [19], p. 266).

ⓓ Unlike ⓑ in U_ε let us choose an arbitrary v and any sequence $\overline{v}^r \rightharpoonup v$ now. We have

$$\begin{aligned}&\langle \mathcal{A}_\varepsilon v^r - \mathcal{A}_\varepsilon v, v^r - v\rangle \\ &= \int_\Omega \left(a(x,x/\varepsilon,v^r(x),\nabla_{\mu_\varepsilon} v^r(x)) - a(x,x/\varepsilon,v^r(x),\nabla_{\mu_\varepsilon} v(x))\right) \\ &\qquad \cdot\left(\nabla_{\mu_\varepsilon} v^r(x) - \nabla_{\mu_\varepsilon} v(x)\right)\,\mathrm{d}\mu_\varepsilon(x) \\ &+ \int_\Omega \left(a(x,x/\varepsilon,v^r(x),\nabla_{\mu_\varepsilon} v(x)) - a(x,x/\varepsilon,v(x),\nabla_{\mu_\varepsilon} v(x))\right) \\ &\qquad \cdot\left(\nabla_{\mu_\varepsilon} v^r(x) - \nabla_{\mu_\varepsilon} v(x)\right)\,\mathrm{d}\mu_\varepsilon(x) \\ &- \int_\Omega \left(f(x,x/\varepsilon,v^r(x)) - f(x,x/\varepsilon,v(x))\right)\cdot v(x)\,\mathrm{d}\mu_\varepsilon(x) \\ &+ \int_\Omega f(x,x/\varepsilon,v(x))\cdot\left(v^r(x) - v(x)\right)\,\mathrm{d}\mu_\varepsilon(x) \\ &- \int_{\partial\Omega} \left(g(x,v^r(x)) - g(x,v(x))\right)\cdot v(x)\,\mathrm{d}\sigma(x) \\ &+ \int_{\partial\Omega} g(x,v(x))\cdot\left(v^r(x) - v(x)\right)\,\mathrm{d}\sigma(x)\,.\end{aligned}$$

By (c) $\overline{v}^r \to v$ in $L^p_{\mu_\varepsilon}(\Omega)^n$; thus the limit passage (based on the Hölder inequality) with respect to the continuity of a, f and g from (b) gives

$$\begin{aligned}&\limsup_{r\to\infty} \langle \mathcal{A}_\varepsilon v^r - \mathcal{A}_\varepsilon v, v^r - v\rangle \\ &= \limsup_{r\to\infty} \int_\Omega \left(a(x,x/\varepsilon,v^r(x),\nabla_{\mu_\varepsilon} v^r(x)) - a(x,x/\varepsilon,v^r(x),\nabla_{\mu_\varepsilon} v(x))\right) \\ &\qquad\qquad \cdot\left(\nabla_{\mu_\varepsilon} v^r(x) - \nabla_{\mu_\varepsilon} v(x)\right)\,\mathrm{d}\mu_\varepsilon(x)\,.\end{aligned}$$

Let us suppose $\overline{v}^r \not\rightharpoonup v$ and believe that the inequality (19) is fulfilled. Then using (17) from (d) (the only part of (d) that was not needed in ⓐ) we deduce that

$$\limsup_{r\to\infty} \langle \mathcal{A}_\varepsilon v^r - \mathcal{A}_\varepsilon v, v^r - v\rangle > 0$$

which is in contrary with (19) immediately.

Theorem 14 informs us that a set U_ε^0 of solutions of Problem 13 cannot be empty. Let U be a set of all two-scale limits v of such sequences $\underline{v}_\varepsilon$ with elements from U_ε that $v_\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup v$ and $\nabla_{\mu_\varepsilon} v_\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \nabla v_\star + \nabla_{\ddot{\mu}} v_1$. Let U' be a subset of U (defined in the same way) where moreover $\nabla_{\mu_\varepsilon} v_\varepsilon \twoheadrightarrow v_\star$ is required. For the study of limit behaviour of (16) with $\varepsilon \to 0$ (to avoid divergence sequences of solutions from U_ε^0), using the simplified notation

$$\begin{aligned} \widehat{a}(x,y,v(x)) &:= a(x,y,v(x),\widehat{u}_0(x,y))\,, \\ \widehat{u}_0(x,y) &:= \nabla u(x) + \nabla_\mu \phi^r(x,y) + \delta\phi(x,y)\,, \\ \widehat{u}_\varepsilon(x) &:= \widehat{u}_0(x,x/\varepsilon) \end{aligned} \tag{21}$$

for each $x\in\Omega$ and $y\in\mathrm{Y}$ with any positive δ, $\phi^r \in L^p_\lambda(\Omega, C_\#(\mathrm{Y})^n)$ (r is a positive integer), $\phi \in L^p_\lambda(\Omega, C_\#(\mathrm{Y})^{n\cdot n})$ and $v = u$ or $v = u_\varepsilon$, we shall slightly regulate the choice of "loads" f and g and "strain-stress relations" a with respect to μ_ε:

(e) If $\underline{u}_\varepsilon$ and $\underline{v}_\varepsilon$ are some sequences of elements from U_ε with two-scale limits $u_\star$ and $v_\star$ and and $\underline{\Phi}_\varepsilon$ is some sequence of elements from $L^p_\lambda(\Omega, C_\#(\mathrm{Y})^{n\cdot n})$ with a two-scale limit Φ_0 where $u, v \in H^{1p}_\lambda(\Omega)^n$ and $\Phi_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y})^{n\cdot n})$ then

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_\Omega \widehat{a}(x,x/\varepsilon,u_\varepsilon(x))\cdot \Phi_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) \\ &\qquad = \int_\Omega \int_\mathrm{Y} \widehat{a}(x,y,u(x))\cdot\Phi_0(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x)\,, \\ &\lim_{\varepsilon\to 0} \int_\Omega f(x,x/\varepsilon,u_\varepsilon(x))\cdot v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon(x) \\ &\qquad = \int_\Omega \int_\mathrm{Y} f(x,y,u(x))\,\mathrm{d}\mu(y)\;\cdot v(x)\,\mathrm{d}\lambda(x)\,, \\ &\lim_{\varepsilon\to 0} \int_{\partial\Omega} g(x,u_\varepsilon(x))\cdot v_\varepsilon(x)\,\mathrm{d}\sigma(x) = \int_{\partial\Omega} g(x,u(x))\cdot v(x)\,\mathrm{d}\sigma(x)\,. \end{aligned}$$

The property (e) looks rather difficult to be verified; thus (for illustration) we shall demonstrate how it could be simplified (using sufficient conditions) in special cases: If a, f and g are independent of the third variable explicitly then the first two relations can be seen as simple consequences of Definition 2, only the third one needs $v_\varepsilon \to v$ in $L^p_\sigma(\partial\Omega)^n$ which (except pure Dirichlet problems, favourable for matematicians, but rare in practice) may not be trivial (the properties of extensions of v_ε onto $\partial\Omega$ have to be studied). If (for general a, f and g again) $\underline{\mu}_\varepsilon = \mu = \lambda$ then Lemma 11 results that u_ε has a weak limit in $U = \underline{U}_\varepsilon$ (which is a subspace of $H^{1p}_{\lambda}(\Omega)^n$ here) immediately, hence (if the Sobolev imbedding theorem holds), up to a subsequence, $u_\varepsilon \to u$ in U and (if the trace theorem holds) $u_\varepsilon \to u$ in $L^p_\sigma(\partial\Omega)^n$. (This is even true for a large class of domains Ω without any respect to a, f and g; the geometrical properties of such class are studied in [28], pp. 58 and 219, in great details, including perverse configurations uncovered by standard theorems.) Then (e) can be checked using continuity arguments from (b) only; Lemma 10 can be helpful, too. Also if we only know that (for any reason) u_ε is bounded in $H^{1p}_{\lambda}(\Omega)^n$ then the Eberlein-Shmul'yan theorem implies, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $H^{1p}_{\lambda}(\Omega)^n$ and the same approach can be applied. In more complicated cases simple criteria are not known; nevertheless, we shall try to formulate a general convergence result:

Theorem 15 (*existence result with $\varepsilon \to 0$*). *Let the assumptions of Theorem 14 and (e) be fulfilled. Then the limit process $\varepsilon \to 0$ converts the integral equation (16) from Problem 13 into its limit form*

$$\int_\Omega \int_{\mathrm{Y}} a(x,y,u(x),\nabla u(x) + \nabla_{\ddot{\mu}} u_1(x,y))\,\mathrm{d}\mu(y)\cdot\nabla v(x)\,\mathrm{d}\lambda(x) \\ + \int_\Omega \int_{\mathrm{Y}} f(x,y,u(x))\,\mathrm{d}\mu(y)\cdot v(x)\,\mathrm{d}\lambda(x) + \int_{\partial\Omega} g(x,u(x))\cdot v(x)\,\mathrm{d}\sigma(x) = 0 \tag{22}$$

for all $v \in U'$ which has at least one solution $u \in U$, $u_1 \in L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y})^n)$.

Proof. For the sake of brevity let us introduce the notation

$$\alpha_\varepsilon(x) := a(x, x/\varepsilon, u_\varepsilon(x), \nabla_{\mu_\varepsilon} u_\varepsilon(x))$$

for every $x \in \Omega$. (Since by Theorem 14 $U^0_\varepsilon \neq \emptyset$, the choice of some $u_\varepsilon \in U^0_\varepsilon$ is possible.) Using (a) we obtain in the same way as in ⓒ in the proof of Theorem 14

$$\begin{aligned} |\alpha_\varepsilon(x)|^q &\leq \Big(\beta_a(x,x/\varepsilon) + \gamma_a\left(|u_\varepsilon(x)|^{p-1} + |\nabla_{\mu_\varepsilon} u_\varepsilon(x)|\right)^{p-1}\Big)^q \\ &\leq 2^{q-1}\beta_a^q(x,x/\varepsilon) + 2^{2(q-1)}\gamma_a^q\left(|u_\varepsilon(x)|^p + |\nabla_{\mu_\varepsilon} u_\varepsilon(x)|^p\right) \end{aligned}$$

for λ-a. e. $x \in \Omega$ and consequently

$$\|\alpha_\varepsilon\|^q_{L^q_{\mu_\varepsilon}(\Omega)^{n\cdot n}} \leq 2^{q-1}\left(\|\beta_a\|^q_{L_{\mu_\varepsilon}(\Omega, C_\#(\mathrm{Y}))} + 2^{q-1}\gamma_a\|u_\varepsilon\|^p_{U_\varepsilon}\right). \tag{23}$$

The convergence of $\underline{\mu_\varepsilon}$ (cf. (2)) enables us to find κ_1 and κ_2 in (20) from ⓒ independently of both v and ε; in particular (for $v = u_\varepsilon \in U^0_\varepsilon$) (20) can be written in the form

$$0 = \langle \mathcal{A}_\varepsilon u_\varepsilon, u_\varepsilon\rangle \geq \kappa_1\|u_\varepsilon\|^p_{U_\varepsilon} - \kappa_2$$

showing that both $(\underline{u}_\varepsilon, \mu_\varepsilon)$ and $(\nabla_{\mu_\varepsilon}\underline{u}_\varepsilon, \mu_\varepsilon)$ are bounded (in sense of Definition 1 and Remark 6). Thanks to (23) the same is true for $(\underline{\alpha}_\varepsilon, \mu_\varepsilon)$. According to Lemma 9 and Lemma 12, up to subsequences, then $\underline{u}_\varepsilon \rightharpoonup\!\!\rightharpoonup u_\star$, $\underline{\alpha}_\varepsilon \rightharpoonup\!\!\rightharpoonup \alpha_0$ and

$\nabla_{\mu_\varepsilon}\underline{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup \nabla u_\star + \nabla_{\ddot{\mu}} u_1$ for some $u \in H^{1\,p}_\lambda(\Omega)^n$, $\alpha_0 \in L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y})^{n.n})$ and $u_1 \in L^p_\lambda(\Omega, H^{1\,p}_{\mu\#}(\mathrm{Y})^n)$. Unfortunately no reasonable relation between α_0 and u with u_1 is available now. To investigate it, let us start with the integration of (17) from (d) with $y = x/\varepsilon$, $z = u_\varepsilon(x)$, $\theta = \nabla_{\mu_\varepsilon} u_\varepsilon(x)$ and $\widetilde{\theta} = \widehat{u}_\varepsilon(x)$ over Ω. We receive

$$\int_\Omega \left(a(x, x/\varepsilon, u_\varepsilon(x), \nabla_{\mu_\varepsilon} u_\varepsilon(x)) - a(x, x/\varepsilon, u_\varepsilon(x), \widehat{u}_\varepsilon(x))\right) \cdot \left(\nabla_{\mu_\varepsilon} u_\varepsilon(x) - \widehat{u}_\varepsilon(x)\right)\, \mathrm{d}\mu_\varepsilon(x) \geq 0$$

which with help of (16) gets the form

$$\begin{aligned}
&- \int_\Omega f(x, x/\varepsilon, u_\varepsilon(x)) \cdot u_\varepsilon(x)\, \mathrm{d}\mu - \int_{\partial\Omega} g(x, u_\varepsilon(x)) \cdot u_\varepsilon(x)\, \mathrm{d}\sigma(x) \\
&- \int_\Omega \widehat{a}(x, x/\varepsilon, u_\varepsilon(x)) \cdot \nabla_{\mu_\varepsilon} u_\varepsilon(x)\, \mathrm{d}\mu_\varepsilon(x) \\
&+ \int_\Omega \widehat{a}(x, x/\varepsilon, u_\varepsilon(x)) \cdot \widehat{u}_\varepsilon(x)\, \mathrm{d}\mu_\varepsilon(x) - \int_\Omega \alpha_\varepsilon(x) \cdot \widehat{u}_\varepsilon(x)\, \mathrm{d}\mu_\varepsilon(x) \geq 0\,.
\end{aligned}$$

But the initial three left-hand-side integrals are exactly those from (e), only $v_\varepsilon = u_\varepsilon$ and also $\Phi_\varepsilon = \nabla_{\mu_\varepsilon} u_\varepsilon$ in the third one had to be set. Since $\widehat{\underline{u}}_\varepsilon \twoheadrightarrow \widehat{u}_0$ (by Remark 5), the same can be repeated with $\Phi_\varepsilon = -\widehat{u}_\varepsilon$ for the fourth one; for the last one $\alpha_\varepsilon \rightharpoonup\!\!\!\rightharpoonup \alpha_0$ has been derived yet. In this way we obtain

$$\begin{aligned}
&- \int_\Omega \int_{\mathrm{Y}} f(x, y, u(x)) \cdot u(x)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) - \int_{\partial\Omega} g(x, u(x)) \cdot u(x)\, \mathrm{d}\sigma(x) \\
&- \int_\Omega \int_{\mathrm{Y}} \widehat{a}(x, y, u(x)) \cdot \left(\nabla u(x) + \nabla_{\ddot{\mu}} u_1(x, y)\right)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) \\
&+ \int_\Omega \int_{\mathrm{Y}} \left(\widehat{a}(x, y, u(x)) - \alpha_0(x, y)\right) \cdot \widehat{u}_0(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) \geq 0
\end{aligned}$$

and in another order (using δ, ϕ and ϕ^r)

$$\begin{aligned}
&- \int_\Omega \int_{\mathrm{Y}} f(x, y, u(x)) \cdot u(x)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) - \int_{\partial\Omega} g(x, u(x)) \cdot u(x)\, \mathrm{d}\sigma(x) \\
&- \int_\Omega \int_{\mathrm{Y}} \alpha_0(x, y)\, \mathrm{d}\mu(y) \cdot \nabla u(x)\, \mathrm{d}\lambda(x) \\
&- \int_\Omega \int_{\mathrm{Y}} \alpha_0(x, y) \cdot \nabla_\mu \phi^r(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) \\
&+ \int_\Omega \int_{\mathrm{Y}} \widehat{a}(x, y, u(x)) \cdot \nabla_{\ddot{\mu}} \left(\phi^r(x, y) - u_1(x, y)\right)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) \\
&+ \delta \int_\Omega \int_{\mathrm{Y}} \left(\widehat{a}(x, y, u(x)) - \alpha_0(x, y)\right) \cdot \phi(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) \geq 0\,.
\end{aligned} \tag{24}$$

But by Definition 2 and (16) (with respect to (e) again) we have

$$
\begin{aligned}
&\int_\Omega \int_\mathrm{Y} f(x,y,u(x)) \cdot u(x)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) + \int_{\partial\Omega} g(x,u(x)) \cdot u(x)\,\mathrm{d}\sigma(x) \\
&+ \int_\Omega \int_\mathrm{Y} \alpha_0(x,y)\,\mathrm{d}\mu(y) \cdot \nabla u(x)\,\mathrm{d}\lambda(x) \\
&= \lim_{\varepsilon\to 0} \int_\Omega f(x,x/\varepsilon,u_\varepsilon(x)) \cdot u(x)\,\mathrm{d}\mu_\varepsilon(x) + \lim_{\varepsilon\to 0} \int_{\partial\Omega} g(x,u_\varepsilon(x)) \cdot u(x)\,\mathrm{d}\sigma(x) \\
&+ \lim_{\varepsilon\to 0} \int_\Omega \alpha_\varepsilon(x) \cdot \nabla u(x)\,\mathrm{d}\mu_\varepsilon(x) = 0
\end{aligned}
$$

and the first, second and third left-hand-side integrals in (24) vanish. For the fourth one we have

$$
\begin{aligned}
&\int_\Omega \int_\mathrm{Y} \alpha_0(x,y) \cdot \nabla_\mu \phi^r(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) \\
&= \lim_{\varepsilon\to 0} \int_\Omega \alpha_\varepsilon(x) \cdot \nabla_{\mu_\varepsilon} \phi^r(x,x/\varepsilon)\,\mathrm{d}\mu_\varepsilon(x)\,,
\end{aligned}
$$

but also

$$
\begin{aligned}
\nabla_{\mu_\varepsilon} \phi^r(x,x/\varepsilon) &= P_{\mu_\varepsilon}(x) \nabla_{\cdot\cdot}\, \phi^r(x,x/\varepsilon) \\
&= \varepsilon \left(P_{\mu_\varepsilon}(x) \nabla \phi^r(x,x/\varepsilon) - P_{\mu_\varepsilon}(x) \nabla_{\cdot}\, \phi^r(x,x/\varepsilon)\right)
\end{aligned}
$$

and (due to the boundedness of $\underline{\alpha}_\varepsilon$) the last limit is zero, too. In particular it is always possible to choose $\overline{\phi}^r \to u_1$ in $L^p_\lambda(\Omega, H^{1p}_{\mu\#}(\mathrm{Y})^n)$ (as in Remark 8); the limit passage $r \to \infty$ then removes the fifth integral. Finally, divided by δ, (24) degenerates to

$$
\begin{aligned}
\lim_{r\to\infty} \int_\Omega \int_\mathrm{Y} \; &(a(x,y,u(x),\nabla u(x) + \nabla_{\ddot{\mu}} \phi^r(x,y) + \delta\phi(x,y)) - \alpha_0(x,y)) \\
&\cdot \phi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) \geq 0
\end{aligned}
$$

which, in particular for $\delta = 1/r$, can be (thanks to the continuity of a from (b)) rewritten as

$$
\int_\Omega \int_\mathrm{Y} \varsigma(x,y) \cdot \phi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) \geq 0 \tag{25}
$$

where

$$
\varsigma(x,y) := a(x,y,u(x),\nabla u(x) + \nabla_{\ddot{\mu}} \phi(x,y)) - \alpha_0(x,y)\,.
$$

Let us consider (consulting Remark 8 again) a sequence $\overline{\varsigma}^r$ of elements from $L^p_\lambda(\Omega, C_\#(\mathrm{Y})^{n.n})$ with the strong limit ς in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y})^{n.n})$. It remains to prove that the norm of each element of this sequence in $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y})^{n.n})$ is zero. Indeed, if this is true then the norm of ς must be zero, too, and

$$
\alpha_0(x,y) = a(x,y,u(x),\nabla u(x) + \nabla_{\ddot{\mu}} u_1(x,y))
$$

for λ-a. e. $x \in \Omega$ and for μ-a. e. $y \in \mathrm{Y}$ which yields (22). Applying the famous Minty trick (cf. [20], p. 261)

$$\phi(x,y) = \epsilon|\varsigma^r(x,y)|^{p-1}\operatorname{sgn}\varsigma^r(x,y)$$

with $\epsilon \in \{-1,1\}$ to (25), we obtain

$$\epsilon \int_\Omega \int_\mathrm{Y} |\varsigma^r(x,y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\lambda(x) \geq 0$$

which (independently of r) forces ς^r to be the zero point of $L^p_\lambda(\Omega, L^p_{\mu\#}(\mathrm{Y})^n)$. Now it is easy to finish this proof: ¿From the just verified two-scale convergence of $\underline{\alpha}_\varepsilon$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0} &\int_\Omega a(x, x/\varepsilon, u_\varepsilon(x), \nabla_{\mu_\varepsilon} u_\varepsilon(x)) \cdot \nabla_{\mu_\varepsilon} v(x)\,\mathrm{d}\mu_\varepsilon(x) \\ &= \int_\Omega \int_\mathrm{Y} a(x, y, u(x), \nabla u(x) + \nabla_{\ddot{\mu}} u_1(x,y))\,\mathrm{d}\mu(y) \cdot \nabla v(x)\,\mathrm{d}\lambda(x)\end{aligned}$$

which, making use of Lemma 10, demonstrates how the first left-hand-side additive term from (16) tends to the corresponding one from (22) if $\varepsilon \to 0$. The same for the second and third terms follows from the second and third equations of (e) (with v_ε unchanged).

References

1. Adams R. A., *Sobolev Spaces*, Academic Press London, 1975.
2. Allaire G., *Homogenization and Two-Scale Convergence*, SIAM J. Math. Anal., **23**, No 6, 1992, 1482-1512.
3. Allaire G., Damlamian A., Hornung U., *Two-Scale Convergence on Periodic Surfaces and Applications*, in Proc. International Conference on Mathematical Modelling of Flow Through Porous Media (World Scientific Pub., Singapore), 1995, 15-25.
4. Arbogast T., Douglas J., Hornung V., *Derivation of the Double Porosity Model of Single Phase Flow via Homogenization Theory*, SIAM J. Math. Anal., **21**, No 4, 1990, 823-836.
5. Babuška I., *Homogenization Approach in Engineering*, in Lecture Notes in Economics and Mathematical Systems (Berkmann M., Kunzi H. P., eds.), Springer, Berlin, 1975, 137-153.
6. Babuška I., *Verification and Validation in Computational Mechanics – Some Mathematical Aspects*, in Equadiff 10 Abstracts (Acad. Sci. Czech Rep., Prague), 2001, 14.
7. Bensoussan A., Lions J. L., Papanicolaou G., *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam, 1978.
8. Bouchité G., Fragalà I., *Homogenization of Thin Structures by Two-Scale Method with Respect to Measures*, SIAM J. Math. Anal. **32**, No 6, 2001, 1198-1226.
9. Bourgeat A., Luckhaus S., Mikelic A., *Convergence of the Homogenization Process for a Double-Porosity Model of Immiscible Two-Phase Flow*, SIAM J. Math. Anal. **27**, No 6, 1996, 1520-1543.

10. Cioranescu D., Donato P., *An Introduction to Homogenization*, Oxford University Press, Oxford, 2000.
11. Clark G. W., Showalter R. W., *Two-Scale Convergence of a Model For Flow in a Partially Fissured Medium*, Electr. J. Diff. Eq., **1999**, No 2, 1999, 1-20.
12. Dalík J., Daněček J., Šťastník S., *A Model of Simultaneous Distribution of Humidity and Temperature in Porous Materials*, Ceramics (Silikáty) **41**, No 2, 1997, 41-46.
13. Dal Maso G., *An Introduction to* Γ*-Convergence*, Birkhäuser, Boston, 1993.
14. Defranceschi A., *An Introduction to Homogenization and G-Convergence*, Int. Center Theor. Physics, Trieste, 1993.
15. De Giorgi E., *Some Remarks on* Γ*-Convergence and Least Square Methods*, in Composite Media and Homogenization Theory (Dal Maso G., Dell'Antonio G.F., eds.), Birkhäuser, Boston, 1991, 135-141.
16. Del Piero G., Owen D. R., *Structured Deformations of Continua*, Arch. Rat. Mech. Anal. **124**, No 2, 1992, 99-155.
17. Fragalà I., *Lower Semicontinuity of Multiple* μ*-Quasiconvex Integrals*, Preprint, Scuola Normale Superiore Pisa, 2001.
18. Fragalà I., Mantegazza C., *On Some Notions of Tangent Space to a Measure*, Proc. Roy. Soc. Edinburgh **129A**, 1999, 331-342.
19. Franců J., *Monotone Operators – A Survey Directed to Applications to Differential Equations*, Appl. Math. **35**, No 4, 1990, 257-301.
20. Fučík S., Kufner A., *Nonlinear Differential Equations* Elsevier, Amsterdam, 1980.
21. Habala P., Hájek P., Zizler V., *Introduction to Banach Spaces*, Matfyzpress, Prague, 1996.
22. Holmbom A., *Homogenization of Parabolic Equations – An Alternative Approach and Some Corrector-Type Results*, Appl. Math. **47**, No 5, 1997, 321-343.
23. *Mean Quadratic Convergence of Signed Random Measures*, Comment. Math. Univ. Carolinae **32**, No 1, 1991, 119-123.
24. Kufner A., John O., Fučík S., *Function Spaces*, Academia, Prague, 1977.
25. Mascarenhas M. L., Toader A.-M., *Scale Convergence in Homogenization*, Preprint, Universidade de Lisboa, 2000.
26. Matache A.-M., Babuška I., *Generalized p-FEM in Homogenization*, Num. Math. **22**, No 1, 2000, 1-33.
27. Matache A.-M., Schwab Ch., *Two-Scale Finite Element Method for Homogenization Problems*, in Seminar of Applied Mathematics, Rep. No. 99-18 (ETH-Zentrum, Zürich), 2001.
28. Maz'ya V. G., *Spaces of S. L. Sobolev* (in Russian), Izdateľstvo Leningradskogo universiteta, Leningrad (St. Petersburg) 1985.
29. Murat F., Tartar L., *Variational Calculus and Homogenization* (in French), Publ. du Laboratoire d'Analyse Numérique, No. 84012 (Univ. Pierre et Marie Curie, Paris), 1978.
30. Nechvátal L., *On Two-Scale Convergence*, in Equadiff 10 Abstracts (Acad. Sci. Czech Rep., Prague), 2001, 149-150.
31. Nguetseng G., *A General Convergence Result for a Functional Related to the Theory of Homogenization*, SIAM J. Math. Anal. **20**, No 3, 1989, 608-623.
32. Roubíček T., *Relaxation in Optimization Theory and Variational Calculus*, Walter de Gruyter, Berlin, 1997.
33. Sanchez-Palencia E., *Nonhomogenous Media and Vibration Theory*, Lecture Notes in Physics 127, Springer, Berlin, 1980.
34. Spagnolo S., *On the Convergence of Solutions of Parrabolic and Elliptic Equations* (in Italian), Ann. Scuola Normale Superiore Pisa, No 22, 1968.

35. Svoboda J., Vala J., *Micromodelling of Creep in Composites with Perfect Matrix / Particle Interfaces*, Kovové materiály (Metallic Materials, Slovak Rep.) **36**, No 2, 1998, 109-126.
36. Vala J., *On One Approach to Numerical Modelling of Creep Flow in Crystalline Materials*, in Proc. Prague Mathematical Conference (Acad. Sci. Czech Rep.), 1996, 325-330.
37. Vala J., *On One Mathematical Model of Creep in Superalloys*, Appl. Math., **43**, No 5, 1998, 351-380.
38. Vala J., *On Mathematical Modelling of Heat and Moisture Transfer through a Porous Medium* (in Czech), in Proc. Programy a algoritmy numerické matematiky 10 (Acad. Sci. Czech Rep., L. Libverda), 2000, 188-194.
39. Vala J., *Two-Scale Limits in Some Nonlinear Problems of Engineering Mechanics*, in Modelling 2001 Abstracts (Univ. West Bohemia, Pilsen), 2001, 65-66, and Proc., to appear.
40. Vala J., Kozák V., Svoboda J., Čadek J., *Modelling Discontinuous Metal Matrix Composite Under Creep Condition: Effect of Diffusional Matter Transport and Interface Sliding*, Scripta Metall. Mater. **30**, No 9, 1994, 1201-1206.
41. Yosida K., *Functional Analysis* (in Russian), Mir, Moscow, 1967.
42. Zhikov V. V., *Connectedness and Homogenization – Examples of Fractal Conductivity*, Mat. Sbornik **187**, No 8, 1996, 3-40.
43. Zhikov V. V., Kozlov S. M., Oleĭnik O. A., *Homogenization of Differential Operators and Integral Functionals*, Springer, Berlin, 1996.
44. Ziemer W., *Weakly Differentiable Functions*, Springer, Berlin, 1989.