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Singular Solutions of the Briot-Bouquet Type Partial Differential Equations

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Abstract. In 1990, Gérard-Tahara [2] introduced the Briot-Bouquet type partial differential equation $t \partial_t u = F(t, x, u, \partial_x u)$, and they determined the structure of singular solutions provided that the characteristic exponent $\rho(x)$ satisfies $\rho(0) \notin \{1, 2, \ldots\}$. In this paper the author determines the structure of singular solutions in the case $\rho(0) \in \{1, 2, \ldots\}$.

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1 Introduction

In this paper, we will study the following type of nonlinear singular first order partial differential equations:

$$t \partial_t u = F(t, x, u, \partial_x u)$$

where $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{C}_t \times \mathbb{C}^n_x$, $\partial_x u = (\partial_1 u, \ldots, \partial_n u)$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$, and $F(t, x, u, v)$ with $v = (v_1, \ldots, v_n)$ is a function defined in a polydisk $\triangle$ centered at the origin of $\mathbb{C}_t \times \mathbb{C}^n_x \times \mathbb{C}_u \times \mathbb{C}^n_v$. Let us denote $\triangle_0 = \triangle \cap \{t = 0, u = 0, v = 0\}$.

The assumptions are as follows:

(A1) $F(t, x, u, v)$ is holomorphic in $\triangle$,
(A2) $F(0, x, 0, 0) = 0$ in $\triangle_0$,
(A3) $\frac{\partial F}{\partial v_i}(0, x, 0, 0) = 0$ in $\triangle_0$ for $i = 1, \ldots, n$.

This is the final form of the paper.
Definition 1. ([2], [3]) If the equation (1) satisfies (A1), (A2) and (A3) we say that the equation (1) is of Briot-Bouquet type with respect to \( t \).

Definition 2. ([2], [3]) Let us define

\[
\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0),
\]

then the holomorphic function \( \rho(x) \) is called the characteristic exponent of the equation (1).

Let us denote by

1. \( \mathcal{R}(\mathbb{C}\setminus\{0\}) \) the universal covering space of \( \mathbb{C}\setminus\{0\} \),
2. \( S_0 = \{ t \in \mathcal{R}(\mathbb{C}\setminus\{0\}); |\arg t| < \theta \} \),
3. \( S(\epsilon(s)) = \{ t \in \mathcal{R}(\mathbb{C}\setminus\{0\}); 0 < |t| < \epsilon(\arg t) \} \) for some positive-valued function \( \epsilon(s) \) defined and continuous on \( \mathbb{R} \),
4. \( D_R = \{ x \in \mathbb{C}^n; |x_i| < R \text{ for } i = 1, \ldots, n \} \),
5. \( \mathbb{C}\{x\} \) the ring of germs of holomorphic functions at the origin of \( \mathbb{C}^n \).

Definition 3. We define the set \( \tilde{O}_+ \) of all functions \( u(t, x) \) satisfying the following conditions;
1. \( u(t, x) \) is holomorphic in \( S(\epsilon(s)) \times D_R \) for some \( \epsilon(s) \) and \( R > 0 \),
2. there is an \( a > 0 \) such that for any \( \theta > 0 \) and any compact subset \( K \) of \( D_R \)

\[
\max_{x \in K} |u(t, x)| = O(|t|^a) \quad \text{as} \quad t \to 0 \quad \text{in} \quad S_\theta.
\]

We know some results on the equation (1) of Briot-Bouquet type with respect to \( t \). We concern the following result. Gérard R. and Tahara H. studied in [2] the structure of holomorphic and singular solutions of the equation (1) and proved the following result;

**Theorem 4 (Gérard R. and Tahara H.).** If the equation (1) is Briot-Bouquet type and \( \rho(0) \notin \mathbb{N}^* = \{1, 2, 3, \ldots\} \) then we have;

1. (Holomorphic solutions) The equation (1) has a unique solution \( u_0(t, x) \) holomorphic near the origin of \( \mathbb{C} \times \mathbb{C}^n \) satisfying \( u_0(0, x) \equiv 0 \).
2. (Singular solutions) Denote by \( S_+ \) the set of all \( \tilde{O}_+ \)-solutions of (1).

\[
S_+ = \begin{cases} 
\{u_0(t, x)\} & \text{when } \Re \rho(0) \leq 0, \\
\{u_0(t, x)\} \cup \{U(\varphi); 0 \neq \varphi(x) \in \mathbb{C}\{x\}\} & \text{when } \Re \rho(0) > 0,
\end{cases}
\]

where \( U(\varphi) \) is an \( \tilde{O}_+ \)-solution of (1) having an expansion of the following form:

\[
U(\varphi) = \sum_{i \geq 1} u_i(x)t^i + \sum_{i+2j+k \geq 1, j \geq 1} \varphi_{i,j,k}(x)t^{i+j\rho(x)}(\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).
\]


In the case $\rho(0) \in \mathbb{N}^*$, Yamane [7] showed that the equation (1) has a holomorphic solution in a region $\{(t, x) \in \mathbb{C} \times \mathbb{C}^n; \ |x| < c|t|^d \leq 1\}$ for some $c > 0$ and $d > 0$, but the solution is not in $S_+$.

The purpose of this paper is to determine $S_+$ in the case $\rho(0) \in \mathbb{N}^*$.

The following main result of this paper is;

**Theorem 5.** If the equation (1) is Briot-Bouquet type and if $\rho(0) = N \in \mathbb{N}^*$ and $\rho(x) \not\equiv \rho(0)$, then

$$S_+ = \{U(\varphi); \quad \varphi(x) \in \mathbb{C}\{x\}\},$$

where $U(\varphi)$ is an $\widetilde{O}_+$-solution of (1) having an expansion of the following form:

$$U(\varphi) = u_1^0(t) + u_0^0(x)\phi_N(t, x) + \sum_{i+|\beta| \geq 2, |\beta| < \infty, |\beta| \leq i+|\beta|-2} u_i^\beta(x) t^i \Phi_N^\beta + w_0^0(x) t^{\rho(x)} + \sum_{i+j+|\beta| \geq 2, k \leq i+|\beta|_0+|\beta|_1} w_{i,j,k}^\beta(x) t^{i+j \rho(x)} \{\log t\}^k \Phi_N^\beta,$$

where $u_N^0(x) \equiv 0$, $w_{0,1,0}^0(x) = \varphi(x)$ is arbitrary holomorphic function and the other coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk, and

$$l = (l_1, \ldots, l_n) \in \mathbb{N}^n, \quad |l| = l_1 + \cdots + l_n, \quad \beta = (\beta_l \in \mathbb{N}; \quad l \in \mathbb{N}^n),$$

$$|\beta| = \sum_{|l| \geq 0} \beta_l, \quad |\beta|_p = \sum_{|l| = p} \beta_l \text{ for } p \geq 0, \quad |\beta|_* = \sum_{|l| \geq 1} (|l| - 1) \beta_l,$$

$$\Phi_N^\beta = \prod_{|l| \geq 0} \left( \frac{\partial_l^l \phi_N}{l!} \right)^{\beta_l}, \quad \partial_x^l = \partial_t^{l_1} \cdots \partial_n^{l_n}, \quad \phi_N(t, x) = \frac{t^\rho(x) - t^N}{\rho(x) - N}.$$

The following lemma will play an important role in the proof of Theorem 5.

At first, we define some notations. We denote for $l \in \mathbb{N}^n$, $e_l = (\beta_k; \quad k \in \mathbb{N}^n)$ with $\beta_1 = 1$ and $\beta_k = 0$ for $k \neq l$ and for $p \in \mathbb{N}$, $e(p) = (i_1, \ldots, i_n)$ with $i_p = 1$ and $i_q = 0$ for $q \neq p$, and denote that $l^1 < l^0$ is defined by $|l^1| < |l^0|$ and $l_i^1 \leq l_i^0$ for $i = 1, \ldots, n$.

**Lemma 6.** Let $\rho(x)$, $\phi_N$ and $\Phi_N^\beta$ be in Theorem 5. Then we have;

1. $\partial_p \Phi_N^\beta = \sum_{|l| \geq 0} \beta_l (l_p + 1) \phi_N^{\beta - e_l + e_l + e(p)}$ for $i = 1, \ldots, n$,
2. $t \partial_t \phi_N = \rho(x) \phi_N + t^N$,
3. $t \partial_t \Phi_N^\beta = |\beta| \rho(x) \Phi_N^\beta + \beta_0 t^N \Phi_N^{\beta - e_0} + \sum_{|l| \geq 1} \sum_{l^1 < l^0} \beta_0 \frac{t^{l_0-1} \rho(x)}{l_0 - l} \Phi_N^{\beta - e_0 + e_1}$.

Proof.

1. By $\partial_p (\partial_x^l \phi_N / l!)^{\beta_l} = \beta_l (\partial_x^l \phi_N / l!)^{\beta_l - 1} \partial_x^{l+e(p)} \phi_N / l!$, we have the result 1.
2. By \( t \partial_t \phi_N = (\rho(x)t^\rho(x) - Nt^N)/(\rho(x) - N) \), we have the result 2.
3. By 2, we have

\[
t \partial_t \left( \frac{\partial_t \phi_N}{l!} \right)^{\beta_l} = \beta_l \left( \frac{\partial_t \phi_N}{l!} \right)^{\beta_l-1} \frac{\partial_t (\rho(x)\phi_N + t^N)}{l!}.
\]

Therefore we have

\[
t \partial_t \left( \frac{\partial_t \phi_N}{l!} \right)^{\beta_l} = \begin{cases} 
\beta_0 \rho(x)\phi_N^{\beta_0} + \beta_0 t^N \phi_N^{\beta_0-1} & \text{if } l = 0 \\
\beta_l \phi(x) \left( \frac{\partial_t \phi_N}{l!} \right)^{\beta_l} + \sum_{0 \leq l < |l|} \beta_l \frac{\partial_t^{l-l} \rho(x)}{(l-l)!} \frac{\partial_t^{|l|} \phi_N}{l!} \left( \frac{\partial_t \phi_N}{l!} \right)^{\beta_l-1} & \text{if } |l| > 0.
\end{cases}
\]

Hence we have the desired result. Q.E.D.

2 Construction of formal solutions in the case \( \rho(0) = 1 \)

By [2] (Gérard-Tahara), if the equation (1) is of Briot-Bouquet type with respect to \( t \), then it is enough to consider the following equation:

\[
Lu = t \partial_t u - \rho(x)u = a(x)t + G_2(x)(t, u, \partial_x u)
\]

where \( \rho(x) \) and \( a(x) \) are holomorphic functions in a neighborhood of the origin, and the function \( G_2(x)(t, X_0, X_1, \ldots, X_n) \) is a holomorphic function in a neighborhood of the origin in \( \mathbb{C}_x^n \times \mathbb{C}_t \times \mathbb{C}_{X_0} \times \mathbb{C}_{X_1} \times \cdots \times \mathbb{C}_{X_n} \) with the following expansion:

\[
G_2(x)(t, X_0, X_1, \ldots, X_n) = \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \{X_0\}^{\alpha_0} \{X_1\}^{\alpha_1} \cdots \{X_n\}^{\alpha_n}
\]

and we may assume that the coefficients \( \{a_{p,\alpha}(x)\}_{p+|\alpha| \geq 2} \) are holomorphic functions on \( D_R \) for a sufficiently small \( R > 0 \). We put \( A_{p,\alpha}(R) := \max_{x \in D_R} |a_{p,\alpha}(x)| \) for \( p + |\alpha| \geq 2 \). Then for \( 0 < r < R \)

\[
\sum_{p+|\alpha| \geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t^p X_0^{\alpha_0} X_1^{\alpha_1} \cdots X_n^{\alpha_n}
\]

is convergent in a neighborhood of the origin.

In this section, we assume \( \rho(0) = 1 \) and \( \rho(x) \neq 1 \) and we will construct formal solutions of the equation (9).

Proposition 7. If \( \rho(0) = 1 \) and \( \rho(x) \neq 1 \), the equation (9) has a family of formal solutions of the form:

\[
u = u_0^\alpha(x) \phi_1 + \sum_{m \geq 2} \sum_{i+|\beta|=m \atop |\beta| \leq m-2} u_i^\beta(x) t^i \Phi_1^\beta + w_{0,1,0}^\rho(x) t^\rho(x) + \sum_{m \geq 2} \sum_{i+j+|\beta|=m \atop j \geq 1, |\beta| \leq m-2} \sum_{k \leq i+|\beta|_0+|\beta|_1} w_{i,j,k}^\beta(x) t^{i+j} \rho(x) \{\log t\}^k \Phi_1^\beta
\]

where \( w_{0,1,0}^0(x) \) is an arbitrary holomorphic function and the other coefficients \( u_i^\beta(x), w_{i,j,k}^\beta(x) \) are holomorphic functions determined by \( w_{0,1,0}^0(x) \) and defined in a common disk.

**Remark 8.** By the relation \(|\beta|_* \leq m - 2\) in summations of the above formal solution, we have \( \beta_l = 0 \) for any \( l \in \mathbb{N}^n \) with \(|l| \geq m\).

We define the following two sets \( U_m \) and \( W_m \) for \( m \geq 1 \) to prove Proposition 7.

**Definition 9.** We denote by \( U_m \) the set of all functions \( u_m \) of the following forms:

\[
\begin{align*}
    u_1 &= u_1^0(x) t + u_0^0(x) \phi_1, \\
    u_m &= \sum_{i+|\beta|=m} u_i^\beta(x) t^i \Phi_{1}^\beta \text{ for } m \geq 2,
\end{align*}
\]

(13)

and denote by \( W_m \) the set of all functions \( w_m \) of the following forms:

\[
\begin{align*}
    w_1 &= w_{0,1,0}^0(x) t^{\rho(x)}, \\
    w_m &= \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta|_0+|\beta|_1} \sum_{j \geq 1, |\beta|_* \leq m - 2} w_{i,j,k}^\beta(x) t^{i+j} \{\log t\}^k \Phi_{1}^\beta \text{ for } m \geq 2
\end{align*}
\]

where \( u_i^\beta(x), w_{i,j,k}^\beta(x) \in \mathbb{C}\{x\} \).

We can rewrite the formal solution (12) as follows:

\[
    u = \sum_{m \geq 1} (u_m + w_m) \text{ where } u_m \in U_m, \ w_m \in W_m.
\]

(14)

Let us show important relations of \( u_m \) and \( w_m \) for \( m \geq 2 \). By Lemma 6, we have

\[
\begin{align*}
    \partial_p u_m &= \sum_{i+|\beta|=m} \left\{ \partial_p u_i^\beta(x) t^i \Phi_{1}^\beta + \sum_{|l|=0}^{m-1} (l_p + 1) \beta_l u_i^\beta(x) t^i \Phi_{1}^{\beta-e_l+e_{l+\rho(p)}} \right\}, \\
    \partial_p w_m &= \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta|_0+|\beta|_1} \sum_{j \geq 1, |\beta|_* \leq m - 2} \left\{ \partial_p w_{i,j,k}^\beta(x) t^{i+j} \{\log t\}^k \Phi_{1}^\beta \right. \\
    &\quad \left. + j \partial_p \rho(x) w_{i,j,k}^\beta(x) t^{i+j} \{\log t\}^{k+1} \Phi_{1}^\beta \right. \\
    &\quad \left. + \sum_{|l|=0}^{m-1} \sum_{|l|} (l_p + 1) \beta_l w_{i,j,k}^\beta(x) t^{i+j} \{\log t\}^k \Phi_{1}^{\beta-e_l+e_{l+\rho(p)}} \right\}
\end{align*}
\]

(15)
for $p = 1, \ldots, n$, and we have

$$Lu_m = \sum_{i+|\beta| = m \atop |\beta| \leq m-2} \{ \{i + (|\beta| - 1)\rho(x)\} u_i^\beta(x) t^i \Phi_1^\beta + \beta_0 u_i^\beta(x) t^{i+1} \Phi_1^{\beta-e_0} \} (16) + \sum_{|l^0| = 1 \atop l^1 < l^0} \sum_{\beta} \beta_{l^0} \frac{\partial_x^{\beta - l^1} \rho(x)}{(l^0 - l^1)!} u_i^\beta(x) t^i \Phi_1^{\beta-e_0 + e_1} \}.$$

$$Lw_m = \sum_{i+j+|\beta| = m \atop j \geq 1, |\beta| \leq m-2} \sum_{\beta} \{ \{i + (j + |\beta| - 1)\rho(x)\} \times w_{i,j,k}^\beta(x) t^{i+j} \rho(x) \{ \log t \}^{k-1} \Phi_1^\beta + \beta_0 w_{i,j,k}^\beta(x) t^{i+j} \rho(x) + 1 \{ \log t \}^{k} \Phi_1^{\beta-e_0}$$

$$+ \sum_{|l^0| = 1 \atop l^1 < l^0} \sum_{\beta} \beta_{l^0} \frac{\partial_x^{\beta - l^1} \rho(x)}{(l^0 - l^1)!} w_{i,j,k}^\beta(x) t^{i+j} \rho(x) \{ \log t \}^{k} \Phi_1^{\beta-e_0 + e_1} \}.$$

We show two lemma.

**Lemma 10.** If $u_m \in U_m$ and $w_m \in W_m$, then $Lu_m \in U_m$ and $Lw_m \in W_m$.

Proof. We prove $Lu_m \in U_m$. We will see all powers of each terms in (16). For the second term in (16), we have $i+1+|\beta| - e_0 = i + |\beta| = m$ and $|\beta| - e_0 = |\beta| \leq m-2$.

For the third term, we have $i+|\beta| - e_0 + e_1 = i + |\beta| = m$ and $|\beta| - e_0 + e_1 = |\beta| (if |l^0| = 1), = |\beta| - (|l^0| - 1)$ (if $|l^0| > 1$ and $|l^1| \leq 1), = |\beta| - |l^0| + |l^1| (if |l^0| > 1$ and $|l^1| > 1$). Further by $l^1 < l^0$, we have $|\beta| - e_0 + e_1 \leq |\beta| \leq m-2$. Hence we have $Lu_m \in U_m$.

We can prove $Lw_m \in W_m$ as $Lu_m \in U_m$, and we omit the details. Q.E.D.

**Lemma 11.** If $u_m \in U_m$ and $w_m \in W_m$, then the following relations hold by the relation (15) for $i, j = 1, \ldots, n$

1. $a(x)U_m \subset U_m$ and $a(x)W_m \subset W_m$ for any holomorphic function $a(x)$,
2. $tU_m$, $\phi_1 U_m \subset U_{m+1}$ and $t^p(x)U_m$, $tW_m$, $t^p(x)W_m$, $\phi_1 W_m \subset W_{m+1}$,
3. $u_m \times u_n$, $\partial_i u_m \times \partial_j u_n$, $\partial_i u_m \times u_n \in U_{m+n}$,
4. $w_m \times w_n$, $\partial_i w_m \times \partial_j w_n$, $\partial_i w_m \times w_n \in W_{m+n}$,
5. $u_m \times w_n$, $\partial_i u_m \times w_n$, $u_m \times \partial_j w_n$, $\partial_i u_m \times \partial_j w_n \in W_{m+n}$.

Proof. This is verified by the relations (15) and (16) but tedious calculations. We may omit the details. Q.E.D.

Let us show that $u_m$ and $w_m$ are determined inductively on $m \geq 1$. By substituting $\sum_{m \geq 1} (u_m + w_m)$ into (9), we have

$$\sum_{m \geq 1} (1 - \rho(x))u_1^0(x) + u_0^{e_0}(x) = a(x), \quad (17)$$
for $m \geq 2$

$$Lu_m = \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^{n} \partial_j u_{m_j,h_j}, \quad (18)$$

$$Lw_m = \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} (u_{m_0,h_0} + w_{m_0,h_0}) \prod_{j=1}^{n} \partial_j (u_{m_j,h_j} + w_{m_j,h_j}) - \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^{n} \partial_j u_{m_j,h_j}, \quad (19)$$

where $|m_n| = \sum_{i=0}^n m_i(\alpha_i)$ and $m_i(\alpha_i) = m_i + \cdots + m_i(\alpha_i)$ for $i = 0, 1, \ldots, n$.

We take any holomorphic function $\varphi(x) \in \mathbb{C}\{x\}$ and put $w_{0,1,0}^0(x) = \varphi(x)$, and by (17), we put $u_1^0(x) \equiv 0$ and $u_0^0(x) = a(x)$.

For $m \geq 2$, let us show that $u_m$ and $w_m$ are determined by induction. By Lemma 11, the right side of (18) belongs to $U_m$ and the right side of (19) belongs to $W_m$. Further by $m_j, h_j \geq 1$, we have $m_j, h_j < m$ for $h_j = 1, \ldots, \alpha_j$ and $j = 0, \ldots, n$.

Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi^\beta_1$ and $t^{i+j} \rho(x) \{ \log t \}^k \Phi^\beta_1$ respectively for (18) and (19), then put

$$
\{i + (|\beta| - 1)\rho(x)\} u^\beta_i(x) \\
+ (\beta_0 + 1) u^{\beta+e_0}_{i-1}(x) + \sum_{|l^0| = 1}^{m-1} \sum_{0 \leq l < l^0} (\beta_0 + 1) \frac{\partial_l^{0-1} \rho(x)}{(l^0 - l)!} u^\beta_i(x) \medskip
=i \Phi^\beta_1 \{a_{p,\alpha} \}_{2 \leq p+|\alpha| \leq m}, \{u^\beta_i(x)\}_{i+|\beta| < m}
$$

and

$$
\{i + (j + |\beta| - 1)\rho(x)\} w^\beta_{i,j,k}(x) + (k + 1) w^\beta_{i,j,k+1}(x) \\
+ (\beta_0 + 1) w^{\beta+e_0}_{i-1,j,k}(x) + \sum_{|l^0| = 1}^{m-1} \sum_{0 \leq l < l^0} (\beta_0 + 1) \frac{\partial_l^{0-1} \rho(x)}{(l^0 - l)!} w^{\beta+e_0}_{i,j,k}(x) \medskip
=g^\beta_{i,j,k} \{a_{p,\alpha} \}_{2 \leq p+|\alpha| \leq m}, \{u^\beta_i(x)\}_{i+|\beta| < m}, \{w^\beta_{i,j,k}(x)\}_{i+|\beta| < m}
$$

We define an order for the multi indices $(i, \beta)$ and $(i, j, k, \beta)$ to show that $u^\beta_i(x)$ and $w^\beta_{i,j,k}(x)$ are determined by (20) and (21).

**Definition 12.** The relation $(i', \beta') < (i, \beta)$ is defined by the following orders;

1. $i' + |\beta'| < i + |\beta|$.
2. If $i' + |\beta'| = i + |\beta|$, then $i' < i$. 
3. If $i' + |\beta'| = i + |\beta|$ and $i' = i$, then $|\beta'|_0 < |\beta|_0$.
4. If $i' + |\beta'| = i + |\beta|$, $i' = i$, $|\beta'|_0 = |\beta|_0$, ..., $|\beta'|_l = |\beta|_l$, then $|\beta'|_{l+1} < |\beta|_{l+1}$.

   The relation $(i', j', k', \beta') < (i, j, k, \beta)$ is defined by the following orders;
   1. $i' + j' + |\beta'| < i + j + |\beta|$.
   2. If $i' + j' + |\beta'| = i + j + |\beta|$, then $i' < i$.
   3. If $i' + j' + |\beta'| = i + j + |\beta|$ and $i' = i$, then $j' < j$.
   4. If $i' + j' + |\beta'| = i + j + |\beta|$, $i' = i$ and $j' = j$, then $|\beta'|_0 < |\beta|_0$.
   5. If $i' + j' + |\beta'| = i + j + |\beta|$, $i' = i$, $j' = j$, $|\beta'|_0 = |\beta|_0$, ..., $|\beta'|_l = |\beta|_l$, then $|\beta'|_{l+1} < |\beta|_{l+1}$.
   6. If $(i', j', \beta') = (i, j, \beta)$, then $k' > k$.

   For $m \geq 2$, we have $i + (|\beta| - 1)\rho(x) \neq 0$ and $i + (j + |\beta| - 1)\rho(x) \neq 0$ by $\rho(0) = 1$. Therefore all the coefficients $u_i^\beta(x)$ and $w_{i,j,k}^\beta(x)$ are determined in the order of Definition 12. Hence we obtain Proposition 7. Q.E.D.

3 Convergence of the formal solutions in the case $\rho(0) = 1$

In this section, we show that the formal solution (12) converges in $\tilde{O}_+$. 

**Proposition 13.** Let $\gamma$ satisfy $0 < \gamma < 1$ and let $\lambda$ be sufficiently large. Then for any sufficiently small $r > 0$ we have the following result:

   For any $\theta > 0$ there is an $\epsilon > 0$ such that the formal solution (12) converges in the following region:

   $$\{(t, x) \in C_t \times C^*_x; \ |\eta(t, \lambda)t| < \epsilon, \ |\eta(t, \lambda)^2t^\rho(x)| < \epsilon, \ |\eta(t, \lambda)t^\gamma| < \epsilon, \ t \in S_\theta \text{ and } x \in D_r \},$$

   where $\eta(t, \lambda) = \max \{((\log t)/\lambda), 1\}$.

   In this section, we put $w_{i,0,0}^\beta(x) := u_i^\beta(x)$ and $w_{i,0,k}^\beta(x) \equiv 0$ for $k \geq 1$ in the formal solution (12). Then the formal solution (12) is as follows:

   $$u = u_{0,0,0}^\beta(x)\phi_1 + u_{0,1,0}^\beta(x)t^\rho(x) + \sum_{m \geq 2} \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta|_0+|\beta|_1} \sum_{|\beta|_* \leq m-2} \sum_{+2(j-1)} w_{i,j,k}^\beta(x)t^{i+j}\rho(x)\log t^k\Phi_1^\beta.$$  \hspace{1cm} (22)

   Let us define the following set $V_m$ for (22).

   **Definition 14.** We denote by $V_m$ the set of all the functions $v_m$ of the following forms:

   $$v_1 = u_{0,0,0}^\beta(x)\phi_1 + u_{0,1,0}^\beta(x)t^\rho(x),$$

   $$v_m = \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta|_0+|\beta|_1} \sum_{|\beta|_* \leq m-2} \sum_{+2(j-1)} w_{i,j,k}^\beta(x)t^{i+j}\rho(x)\log t^k\Phi_1^\beta \text{ for } m \geq 2.$$  \hspace{1cm} (23)
We define the following estimate for the function \( v_m \).

**Definition 15.** For the function \((23)\), we define

\[
||v_1||_{r,c,\lambda} = ||v_1||_{r,c} := ||w_{0,0,0}^0||_r + ||w_{0,1,0}^0||_r,
\]

\[
||v_m||_{r,c,\lambda} := \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta|_0 + |\beta|_1 + 2(j-1)} ||w_{i,j,k}^\beta||_{r,c} \lambda^k
\quad \text{for } m \geq 2
\]

for \( c > 0 \) and \( \lambda > 0 \), where

\[
||w_{i,j,k}^\beta||_r = \max_{x \in D_r} |w_{i,j,k}^\beta(x)| \quad \text{and} \quad <\beta> = \sum_{|l| \geq 0} (|l| + 1)\beta_l.
\]

We will make use of

**Lemma 16.** For a holomorphic function \( f(x) \) on \( D_R \), we have

\[
||\partial_x^\alpha f||_{R_0} \leq \frac{\alpha!}{(R - R_0)^{|\alpha|}} ||f||_R \quad \text{for } 0 < R_0 < R.
\]

Proof. By Cauchy’s integral formula, we have the desired result, and we omit the details. Q.E.D

**Lemma 17.** If a holomorphic function \( f(x) \) on \( D_R \) satisfies

\[
||f||_{R_0} \leq \frac{C}{(R - r)^p} \quad \text{for } 0 < r < R
\]

then we have

\[
||\partial_i f||_{R_0} \leq \frac{Ce(p + 1)}{(R - r)^{p+1}} \quad \text{for } 0 < r < R, \quad i = 1, \ldots, n.
\]

For the proof, see Hörmander ([5]lemma 5.1.3)

Let us show the following estimate for the function \( Lv_m \).

**Lemma 18.** Let \( 0 < R_0 < R \). Then there exists a positive constant \( \sigma \) such that for \( m \geq 2 \), if \( v_m \in V_m \) we have

\[
||Lv_m||_{r,c,\lambda} \geq \frac{\sigma m}{2} ||v_m||_{r,c,\lambda} \quad \text{for } 0 < r \leq R_0
\]

for sufficiently small \( c > 0 \) and sufficiently large \( \lambda > 0 \).
Proof. Let us give an estimate the second, the third and the fourth term in the right side of the second relation in (16) respectively.

For the second term, since \( k \leq i + |\beta|_0 + |\beta|_1 + 2(j - 1) \leq 2m \) by \( i + j + |\beta| = m \) we have

\[
T_2 := \sum_{i+j+|\beta|=m \atop |\beta|_* \leq m-2} \sum_{k \leq i + |\beta|_0 + |\beta|_1} \frac{|w^{\beta}_{i,j,k+1}|_r \lambda^{k-1}}{c^{<\beta>}} \leq \frac{2m}{\lambda} ||v_m||_{r,c,\lambda}. \tag{30}
\]

For the fourth term, we have

\[
T_4 := \sum_{i+j+|\beta|=m \atop |\beta|_* \leq m-2} \sum_{|\beta|_0+|\beta|_1 \leq l^0 < t_0} \sum_{l^1 < t^0} \frac{\beta_{l_0} ||\partial_x^{l^0-l^1} \rho w^{\beta}_{i,j,k}||_r \lambda^k}{(l^0 - l^1)!} \frac{c^{<\beta>}}{c^{<\beta,>}}. \tag{31}
\]

By Lemma 16, we have

\[
\sum_{l^1 < l^0} c^{|l^0| - |l^1|} \frac{||\partial_x^{l^0-l^1} \rho||_R \lambda^k}{(l^0 - l^1)!} \leq \sum_{l^1 < l^0} \left( \frac{c}{R - R_0} \right)^{|l^0| - |l^1|} \frac{||\rho||_R}{(R^0 - 1)} \leq \frac{cn ||\rho||_R}{R - R_0} \left( \frac{R - R_0}{R - R_0 - c} \right)^n \tag{32}
\]

for sufficiently small \( c > 0 \). Therefore by (31) and (32), we have

\[
T_4 \leq \kappa(c) \sum_{i+j+|\beta|=m \atop |\beta|_* \leq m-2} \sum_{|\beta|_0+|\beta|_1 \leq l^0 < t_0} \sum_{l^1 < t^0} \frac{\beta_{l_0} ||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta,>}}. \tag{33}
\]

where \( \kappa(c) := \frac{cn}{R - R_0} \left( \frac{R - R_0}{R - R_0 - c} \right)^n ||\rho||_R. \)

For the third term, we have

\[
T_3 := \sum_{i+j+|\beta|=m \atop |\beta|_* \leq m-2} \sum_{|\beta|_0+|\beta|_1 \leq l^0 < t_0} \frac{\beta_{l_0} ||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta-\epsilon_0>}} \leq \frac{c \beta_{l_0} ||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta>}}.
\]
Therefore, since \( c\beta_0 + \kappa(c) \sum_{|\beta| = 1}^{m-1} \beta_0 \leq \frac{2}{3} m \) by the conditions \( \kappa(0) = 0 \) and \( i + j + |\beta| = m \geq 2 \) for sufficiently small \( c > 0 \) and some \( \sigma > 0 \) we have

\[
T_2 + T_3 + T_4 \leq \left( \frac{2m}{\lambda} + \frac{\sigma m}{3} \right) ||v_m||_{r,c,\lambda}.
\] (34)

Further we have \( |i + (j + |\beta| - 1)\rho(x)| \geq \sigma m \) by the condition \( \rho(0) = 1 \) and \( i + j + |\beta| = m \geq 2 \). Therefore we have

\[
||Lv_m||_{r,c,\lambda} \geq \left( \sigma m - \frac{2m}{\lambda} - \frac{\sigma m}{3} \right) ||v_m||_{r,c,\lambda}.
\] (35)

Hence for sufficiently small \( c > 0 \) and sufficiently large \( \lambda > 0 \), we obtain the desired result. Q.E.D.

Let us estimate the function \( \partial_i v_m \).

**Definition 19.** For the function \( v_m \in V_m \) we define

\[
D_p v_m := \sum_{i+j+|\beta|=m} \sum_{\beta \leq m} \partial_p w_{i,j,k}(x) t^{i+j+|\beta|} \{\log t\}^k \Phi^\beta
\] (36)

for \( p = 1, \ldots, n \).

**Lemma 20.** If \( v_m \in V_m \), then for \( i = 1, \ldots, n \), we have

\[
||\partial_i v_m||_{r,c,\lambda} \leq ||D_i v_m||_{r,c,\lambda} + c_0 \lambda m ||v_m||_{r,c,\lambda} + \frac{3m - 2}{c} ||v_m||_{r,c,\lambda} \quad \text{for} \quad 0 < r \leq R_0.
\] (37)

Proof. We have

\[
\sum_{|l| \geq 0} (p + 1)\beta_l \leq \sum_{|l| = 0} (p + 1)\beta_l = 2|\beta| + [\beta] \leq 3m - 2.
\] (38)

We put \( c_0 = \max_{i=1,\ldots,n} \{||\partial_i \rho||_{R_0}\} \), and by the relations (15), (38) and \( j \leq m \) we obtain the desired estimate. Q.E.D.

Therefore by the relations (18), (19) and Lemma 18, 20, we have the following lemma.

**Lemma 21.** If \( u = \sum_{m \geq 1} v_m \) is a formal solution of the equation (9) constructing in Section 2, we have the following inequality for \( v_m \) (\( m \geq 2 \)):

\[
||Lv_m||_{r,c,\lambda} \leq \sum_{p+|\alpha| \geq 2} a_{p,\alpha} \prod_{h_0 = 1}^{a_0} ||v_{m_0,h_0}||_{r,c,\lambda}
\]

\[
\times \prod_{i=1}^{n} \prod_{h_i = 1}^{\alpha_i} \left\{ ||D_i v_{m_i,h_i}||_{r,c,\lambda} + c_0 \lambda m_i h_i ||v_{m_i,h_i}||_{r,c,\lambda} + \frac{3m_i h_i - 2}{c} ||v_{m_i,h_i}||_{r,c,\lambda} \right\}.
\]
Let us define a majorant equation to show that the formal solution (22) converges. We take $A_1$ so that
\[
\frac{||w^{0}_{0,0,0}||_R}{c} + ||w^0_{0,1,0}||_R \leq A_1,
\]
\[
\frac{||\partial_i w^{0}_{0,0,0}||_R}{c} + ||\partial_i w^0_{0,1,0}||_R \leq A_1
\]
for $i = 1, \ldots, n$.

Then we consider the following equation:
\[
\frac{\sigma}{2} Y = \frac{\sigma}{2} A_1 t_1 
+ \frac{1}{R - r} \sum_{p + |\alpha| \geq 2} \frac{A_{p,\alpha}(R)}{(R - r)^{p + |\alpha| - 2}} t_1^p Y^\alpha \prod_{i=1}^n \left( eY + c_0 \lambda Y + \frac{3}{c} Y \right)^{\alpha_i}.
\]

The equation (39) has a unique holomorphic solution $Y = Y(t_1)$ with $Y(0) = 0$ at $(Y, t_1) = (0, 0)$ by implicit function theorem. By an easy calculation, the solution $Y = Y(t_1)$ has the following form:
\[
Y = \sum_{m \geq 1} Y_m t_1^m \quad \text{with} \quad Y_m = \frac{C_m}{(R - r)^{m-1}}
\]
where $Y_1 = C_1 = A_1$ and $C_m \geq 0$ for $m \geq 1$.

Then we have;

**Lemma 22.** For $m \geq 1$, we have
\[
m ||v_m||_{r,c,\lambda} \leq Y_m \quad \text{for} \quad 0 < r \leq R_0
\]
\[
||D_i v_m||_{r,c,\lambda} \leq eY_m \quad \text{for} \quad 0 < r \leq R_0,
\]
for $i = 1, \ldots, n$.

Proof. By $A_1 = Y_1$ and the definition of $A_1$, (41) and (42) hold for $m = 1$.

By induction on $m$, let us show that (41) and (42) hold for $m \geq 2$. By substituting the solution $Y = \sum_{m \geq 1} Y_m t_1^m$ into the equation (39), we have the following relation:
\[
\frac{\sigma}{2} Y_m = \frac{1}{R - r} \sum_{p + |\alpha| \geq 2} \frac{A_{p,\alpha}(R)}{(R - r)^{p + |\alpha| - 2}} \prod_{h_0=1}^{\alpha_0} Y_{m_0, h_0} 
\times \prod_{i=1}^n \prod_{h_i=1}^{\alpha_i} \left\{ eY_{m_i, h_i} + c_0 \lambda Y_{m_i, h_i} + \frac{3}{c} Y_{m_i, h_i} \right\}
\]
for \( m \geq 2 \). Therefore if we assume that (41) and (42) hold for \( m_i, h_i < m \), by (43), Lemma 18 and Lemma 21 we obtain

\[
\frac{\sigma}{2} m \| v_m \|_{r,c,\lambda} \leq (R - r) \frac{\sigma}{2} Y_m.
\]

(44)

Therefore we have

\[
m \| v_m \|_{r,c,\lambda} \leq (R - r) Y_m \leq Y_m.
\]

(45)

The relation (45) is rewritten as follows:

\[
m \sum_{i+j+|\beta| = m} \sum_{k \leq m-2 \text{ or } 2(j-1)} \| w_{i,j,k} \|_{r,\lambda^k} \leq C_m \frac{(R - r)^{m-2}}{m}.
\]

(46)

By (46) and Lemma 17, we have

\[
m \| D_i v_m \|_{r,c,\lambda} \leq (m - 1) e C_m \frac{(R - r)^{m-1}}{m-1}
\]

(47)

for \( i = 1, \ldots, n \) and \( 0 < r < R < 1 \). Therefore we have

\[
\| D_i v_m \|_{r,c,\lambda} \leq \frac{e C_m}{(R - r)^{m-1}} = e Y_m.
\]

(48)

Hence (41) and (42) hold for \( m \geq 2 \). Q.E.D.

Let us show that the formal solution (22) converges by using (41) in Lemma 22. We put (22) as follows:

\[
u = u_0 e_0(x) \phi_1 + w_{0,1,0}(x) \rho(x) + \sum_{m \geq 2} \sum_{i+j+|\beta| = m} \sum_{k \leq m-2 \text{ or } 2(j-1)} \frac{w_{i,j,k}(x) \lambda^k}{c^{<\beta>} \rho(x)^{i+j+|\beta|} \lambda^k} t^{i+j+|\beta|} \psi_1^\beta,
\]

where

\[
\psi_1^\beta = \prod_{|l| \geq 0} \left( c^{\beta_l+1} \frac{\partial \phi_1}{l!} \right)^{\beta_l}.
\]

(49)

Firstly let us estimate (49). For \( \| \phi_1 \|_R \), we have the following lemma.

**Lemma 23.** For any \( \gamma \) with \( 0 < \gamma < 1 \), there is an \( R > 0 \) such that

\[
\| \phi_1 \|_R = O(t^{\gamma}) \quad \text{as } t \to 0 \text{ in } S_\theta
\]

(50)

holds for any \( \theta > 0 \).
Proof. We put
\[ \phi_1 = t^\gamma \frac{t^{\rho_0(x) + \alpha} - t^\alpha}{\rho_0(x)} \] (51)
with \( \alpha + \gamma = 1 \) and \( \rho_0(x) = \rho(x) - 1 \). Then we can take \( R > 0 \) with
\[ ||\rho_0||_R < \alpha \] (52)
by \( \rho_0(0) = 0 \). Therefore we have
\[ \left| \left| \frac{t^{\rho_0(x) + \alpha} - t^\alpha}{\rho_0(x)} \right| \right|_R \leq \left| \log t \right| ||t^{\alpha} - ||\rho_0||_R \to 0 \text{ as } t \to 0 \text{ in } S_\theta \] (53)
for and any \( \theta > 0 \). Hence we have the desired result. Q.E.D.

By Lemma 23, there exists a positive constant \( c_1 \) such that
\[ ||\phi_1||_R \leq c_1|t|^\gamma \text{ in } S_\theta. \] (54)

By Lemma 16 and (54), for \( |l| \geq 0 \) we have
\[ ||\partial_x^l \phi_1||_{R_0} \leq \frac{|l|!}{(R-R_0)|l|} |t|^\gamma \leq \frac{|l|!c_1}{(R-R_0)|l|} |t|^\gamma \text{ for } 0 < R_0 < R. \] (55)

Therefore, we have
\[ ||\psi_1^\beta||_{R_0} \leq \prod_{|l| \geq 0} \left( c^{(|l|+1)} \frac{c_1}{(R-R_0)|l|} |t|^\gamma \right)^{\beta_l} \leq \left( \frac{c}{R-R_0} \right)^{<\beta>} (c_1(R-R_0)|t|^\gamma)^{|\beta|} \] (56)
for \( 0 < R_0 < R \) in \( S_\theta \).

Let us estimate \( t^{i+j\rho(x)} \left( \frac{\log t}{\lambda} \right)^k \psi_1^\beta \).

We put \( \eta(t, \lambda) = \max \left\{ \left| \frac{\log t}{\lambda} \right|, 1 \right\} \), \( c_2 = \max \left\{ \frac{c}{R-R_0}, 1 \right\} \) and \( c_3 = c_1(R-R_0) \).

Since we have
\[ <\beta > \leq \beta_1 + \beta_2 \leq i + j + 3|\beta| \] (57)
and
\[ k \leq i + |\beta|_1 + 2(j-1) \leq i + |\beta| + 2j \] (58)
we obtain
\[ \left| \left| t^{i+j\rho(x)} \left( \frac{\log t}{\lambda} \right)^k \psi_1^\beta \right| \right|_r \leq \left\{ \left| c_2 \eta(t, \lambda) t \right| \right\}^i \left\{ \left| c_2 \eta(t, \lambda)^2 t^{\rho(x)} \right|_r \right\}^j \left\{ \left| (c_2)^3 c_3 \eta(t, \lambda) t^{\gamma} \right| \right\}^{|\beta|} \] in \( S_\theta \). For any sufficiently small \( \epsilon > 0 \), there exists a sufficiently small \( |t| \) in \( S_\theta \) such that
\[ |c_2 \eta(t, \lambda) t| < \epsilon, \quad |c_2 \eta(t, \lambda)^2 t^{\rho(x)}|_r < \epsilon, \quad |(c_2)^3 c_3 \eta(t, \lambda) t^{\gamma}| < \epsilon, \] (59)
and we obtain
\[ \left\| t^{i+j\rho(x)} \left( \frac{\log t}{\lambda} \right) \Psi_1^\beta \right\|_r \leq \epsilon^m. \] (60)

Then by Lemma 22, we have
\[ \|u\|_r \leq \sum_{m \geq 1} Y_m \epsilon^m \] (61)

for sufficiently small \(|t|\) in \(S_\theta\). Hence the formal solution (22) converges for \(x \in D_r\) and sufficiently small \(|t|\) in \(S_\theta\). Q.E.D.

4 Completion of the proof of Theorem 5 in the case 
\(\rho(0) = 1\)

In this section, let us complete the proof of Theorem 5 in the case \(\rho(0) = 1\).

We know the following theorem.

**Theorem 24.** If \(u_i(t, x) \in \tilde{O}_+\) (\(i = 1, 2\)) are solutions of (9), we have;

1. For any \(a < \rho(0) - 1\), we have \(t^{-a}(u_1 - u_2) \in \tilde{O}_+\).
2. If \(t^{-b}(u_1 - u_2) \in \tilde{O}_+\) for some \(b \geq \rho(0) = 1\), we have \(u_1(t, x) = u_2(t, x)\) in \(\tilde{O}_+\).

For the proof, see Gérard and Tahara ([2] Theorem 3).

By the discussions in sections 2, 3 and 4, we already know the following results;

(C1) If \(\rho(0) = 1\) and \(\rho(x) \neq 1\), for any \(\varphi(x) \in \mathbb{C}\{x\}\), the equation (1) has a unique \(\tilde{O}_+\)-solution \(U(\varphi)(t, x)\) having an expansion of the form
\[
U(\varphi) = w_{0,0,0}^\varphi(x) \phi_1 + w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{m \geq 2} \sum_{i+|\beta|=m} u_i^\beta(x) t^i \Phi_1^\beta (62)
\]
\[+ \sum_{m \geq 2} \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta|0+|\beta|1} \sum_{j \geq 1, |\beta|* \leq m-2} w_{i,j,k}^\beta(x) t^{i+j+\rho(x)} \{\log t\}^k \Phi_1^\beta \]

with \(w_{0,1,0}^0(x) = \varphi(x)\), where all the coefficients \(u_i^\beta(x), w_{i,j,k}^\beta(x)\) are holomorphic in a common disk centered at the origin of \(\mathbb{C}_x^n\). If we take \(\varphi(x) = 0\), then the solution \(u_0(t, x)\) has the expansion
\[
u_0(t, x) = U(0) = u_0^{e_0}(x) \phi_1 + \sum_{m \geq 2} \sum_{i+|\beta|=m} u_i^\beta(x) t^i \Phi_1^\beta. \] (63)

(C2) If \(\rho(0) = 1\) and \(\rho(x) \neq 1\), and if a solution \(u(t, x) \in \tilde{O}_+\) of the equation (1) is expressed in the form
\[
t^{-1} \left( u(t, x) - u_0^{e_0}(x) \phi_1(t, x) - \varphi(x) t^{\rho(x)} \right) \in \tilde{O}_+, \] (64)
then the coefficient $u_0^e(x)$ is uniquely determined by the equation (1), and they are independent of $\varphi(x)$.

If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, by (C1) we have

$$S_+ \supset \{ U(\varphi); \varphi(x) \in \mathbb{C}\{x\} \}.$$  

(65)

Hence it is sufficient to prove the following proposition to complete the proof of the main theorem.

**Proposition 25.** Assume (A1), (A2) and (A3). Let $u_0(t, x)$ and $U(\varphi)(t, x)$ be as above. If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, then we can find a $\varphi(x) \in \mathbb{C}\{x\}$ such that $u(t, x) \equiv U(\varphi)(t, x)$ holds in $\tilde{O}_+$. 

The proof of this proposition is almost the same as that of Proposition 2 in Gérard and Tahara [1]; so we may omit the details. Q.E.D.

By (65) and Proposition 25 we obtain the main theorem in the case $\rho(0) = 1$ and $\rho(x) \not\equiv 1$. Q.E.D.

**5 Proof of Theorem 5 in the case $\rho(0) = N$**

In Section 2, 3, and 4, we have proved Theorem 5 in the case $\rho(0) = 1$. In this section, we will prove Theorem 5 in the case $\rho(0) = N \geq 2$ and $\rho(x) \not\equiv N$.

We put

$$u(t, x) = \sum_{i=1}^{N-1} u_i(x) t^i + t^{N-1} w(t, x),$$  

(66)

where $u_i(x) \in \mathbb{C}\{x\}$ ($1 \leq i \leq N - 1$) and $w(t, x) \in \tilde{O}_+$.

Then by an easy calculation we see

**Lemma 26.** If the function (66) is a solution of the equation (9), the functions $u_1(x), \ldots, u_{N-1}(x)$ are uniquely determined and $w(t, x)$ satisfies an equation of the following form:

$$(t\partial_t - \rho(x) + N - 1)w = ta(t, x) + tA_0(t, x)w + t \sum_{i=1}^{n} A_i(t, x) \partial_i w$$  

(67)

$$+ \sum_{|\alpha| \geq 2} i^{(N-1)(|\alpha|-1)} A_\alpha(t, x) w^{\alpha_0} \prod_{i=1}^{n} (\partial_i w)^{\alpha_i},$$

where

$$a(t, x) = \frac{1}{t^N} (G_2(x)(t, w_0, \partial_x w_0) + ta(x) - (t\partial_t - \rho(x))w_0)$$  

(68)
with \( w_0 = \sum_{i=1}^{N-1} u_i(x)t^i \) and

\[
A_i(t, x) = \frac{1}{t} \frac{\partial G_2}{\partial x_i}(t, w_0, \partial_x w_0), \quad i = 0, 1, \ldots, n,
\]

\[
A_\alpha(t, x) = \frac{1}{\alpha!} \frac{\partial^{\alpha_1} G_2}{\partial X_\alpha}(t, w_0, \partial_x w_0), \quad |\alpha| \geq 2.
\]

Since the equation \((67)\) satisfies the conditions (A1), (A2), (A3) and the characteristic exponents \( \rho^N(x) = \rho(x) - N + 1 \) satisfies \( \rho^N(0) = 1 \), we can apply the results in sections 2, 3 and 4.

Further, by the form of all the nonlinear parts of the equation \((67)\), we see that the formal solution constructed in Section 2 has the following form:

\[
w = u_0^{N, c_0}(x)\phi_{N,1} + w_0^{N, 0}(x)t^{\rho^N(x)} + \sum_{i \geq 2} u_i^{N}(x)t^i + \sum_{m \geq 2} \sum_{i+|\beta| = m} u_i^{N, \beta}(x)t^{i+(N-1)(|\beta|-1)}\Phi_{N,1}^{\beta} + \sum_{m \geq 2} \sum_{i+j+|\beta| = m} \sum_{k \leq i+|\beta|_0 + |\beta|_1} w_{i, j, k}^{N, \beta}(x)t^{i+j+\rho^N(x)}(\log t)^k\Phi_{N,1}^{\beta},
\]

where \( \Phi_{N,1}^{\beta} = \prod_{|l| \geq 0} \left( \frac{\partial_x \phi_{N,1}}{l!} \right)^{\beta_l} \) and \( \phi_{N,1} = \frac{t^{\rho^N(x)} - t}{\rho^N(x) - 1} \). Therefore we have

\[
u = \sum_{i=1}^{N-1} u_i(x)t^i + u_0^{N, c_0}(x)\phi_N + w_0^{N, 0}(x)t^{\rho(x)} + \sum_{i \geq 2} u_i^{N}(x)t^{i+N-1} + \sum_{m \geq 2} \sum_{i+|\beta| = m} u_i^{N, \beta}(x)t^{i}\Phi_{N}^{\beta} + \sum_{m \geq 2} \sum_{i+j+|\beta| = m} \sum_{k \leq i+|\beta|_0 + |\beta|_1} w_{i, j, k}^{N, \beta}(x)t^{i+j+\rho(x)}(\log t)^k\Phi_{N}^{\beta}.
\]

We put

\[
u_i^{N}(x) \mapsto u_{i+N-1}(x) \quad \text{for} \quad i \geq 2, \quad \nu_i^{N, \beta}(x) \mapsto u_i^{\beta}(x) \quad \text{for} \quad |\beta| \geq 1, \quad w_{i, j, k}^{N, \beta}(x) \mapsto w_i^{\beta}(x) \quad \text{for any} \quad (i, j, k, \beta),
\]

and we have \( u_0^N(x) \equiv 0 \) by the form of the solution \((69)\) and the above relations. Hence this completes the proof of Theorem 5. Q.E.D.
References