EQUADIFF 10

Hiroshi Yamazawa

Singular solutions of the Briot-Bouquet type partial differential equations

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 443--460.

Persistent URL: http://dml.cz/dmlcz/700375

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 443–460

Singular Solutions of the Briot-Bouquet Type Partial Differential Equations

Hiroshi Yamazawa

Department of Language and Culture, Caritas College,
2-29-1 Azamino, Aoba-ku, Yokohama, Japan,
Email: yamazawa@caritas.ac.jp

Abstract. In 1990, Gérard-Tahara [2] introduced the Briot-Bouquet type partial differential equation $t\partial_t u = F(t, x, u, \partial_x u)$, and they determined the structure of singular solutions provided that the characteristic exponent $\rho(x)$ satisfies $\rho(0) \notin \{1, 2, \dots\}$. In this paper the author determines the structure of singular solutions in the case $\rho(0) \in \{1, 2, \dots\}$.

MSC 2000. 35A20, 35C10

Keywords. Singular solutions, Characterirtic exponent

1 Introduction

In this paper, we will study the following type of nonlinear singular first order partial differential equations:

$$t\partial_t u = F\left(t, x, u, \partial_x u\right) \tag{1}$$

where $(t, x) = (t, x_1, \dots, x_n) \in \mathbf{C}_t \times \mathbf{C}_x^n$, $\partial_x u = (\partial_1 u, \dots, \partial_n u)$, $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_i = \dfrac{\partial}{\partial x_i}$ for $i = 1, \dots, n$, and $F(t, x, u, v)$ with $v = (v_1, \dots, v_n)$ is a function defined in a polydisk $\triangle$ centered at the origin of $\mathbf{C}_t \times \mathbf{C}_x^n \times \mathbf{C}_u \times \mathbf{C}_v^n$. Let us denote $\triangle_0 = \triangle \cap \{t = 0, u = 0, v = 0\}$.

The assumptions are as follows:

$$\begin{aligned}
&\text{(A1) } F(t, x, u, v) \text{ is holomorphic in } \triangle, \\
&\text{(A2) } F(0, x, 0, 0) = 0 \text{ in } \triangle_0, \\
&\text{(A3) } \frac{\partial F}{\partial v_i}(0, x, 0, 0) = 0 \text{ in } \triangle_0 \text{ for } i = 1, \dots, n.
\end{aligned} \tag{2}$$

This is the final form of the paper.

Definition 1. ([2], [3]) If the equation (1) satisfies $(A1)$, $(A2)$ and $(A3)$ we say that the equation (1) is of Briot-Bouquet type with respect to t.

Definition 2. ([2], [3]) Let us define

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0), \tag{3}$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1).

Let us denote by

1. $\mathcal{R}(\mathbf{C}\backslash\{0\})$ the universal covering space of $\mathbf{C}\backslash\{0\}$,
2. $S_\theta = \{t \in \mathcal{R}(\mathbf{C}\backslash\{0\});\ |\arg t| < \theta\}$,
3. $S(\epsilon(s)) = \{t \in \mathcal{R}(\mathbf{C}\backslash\{0\});\ 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbf{R}$,
4. $D_R = \{x \in \mathbf{C}^n;\ |x_i| < R \text{ for } i = 1, \ldots, n\}$,
5. $\mathbf{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbf{C}^n$.

Definition 3. We define the set $\widetilde{\mathcal{O}}_+$ of all functions $u(t,x)$ satisfying the following conditions;
1. $u(t,x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,
2. there is an $a > 0$ such that for any $\theta > 0$ and any compact subset K of D_R

$$\max_{x \in K} |u(t,x)| = O\left(|t|^a\right) \quad \text{as} \quad t \to 0 \quad \text{in} \quad S_\theta. \tag{4}$$

We know some results on the equation (1) of Briot-Bouquet type with respect to t. We concern the following result. Gérard R. and Tahara H. studied in [2] the structure of holomorphic and singular solutions of the equation (1) and proved the following result;

Theorem 4 (Gérard R. and Tahara H.). *If the equation (1) is Briot-Bouquet type and $\rho(0) \notin \mathbf{N}^* = \{1, 2, 3, \ldots\}$ then we have;*
(1) (*Holomorphic solutions*) *The equation (1) has a unique solution $u_0(t,x)$ holomorphic near the origin of $\mathbf{C} \times \mathbf{C}^n$ satisfying $u_0(0,x) \equiv 0$.*
(2) (*Singular solutions*) *Denote by S_+ the set of all $\widetilde{\mathcal{O}}_+$-solutions of (1).*

$$S_+ = \begin{cases} \{u_0(t,x)\} & \textit{when} \quad Re\rho(0) \le 0, \\ \{u_0(t,x)\} \cup \{U(\varphi); 0 \ne \varphi(x) \in \mathbf{C}\{x\}\} & \textit{when} \quad Re\rho(0) > 0, \end{cases} \tag{5}$$

where $U(\varphi)$ is an $\widetilde{\mathcal{O}}_+$-solution of (1) having an expansion of the following form:

$$U(\varphi) = \sum_{i \ge 1} u_i(x) t^i + \sum_{i+2j \ge k+2, j \ge 1} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)^k, \ \varphi_{0,1,0}(x) = \varphi(x). \tag{6}$$

In the case $\rho(0) \in \mathbf{N}^*$, Yamane [7] showed that the equation (1) has a holomolphic solution in a region $\{(t,x) \in \mathbf{C} \times \mathbf{C}^n;\ |x| < c|t|^d \ll 1\}$ for some $c > 0$ and $d > 0$, but the solution is not in S_+.

The purpose of this paper is to determine S_+ in the case $\rho(0) \in \mathbf{N}^*$.

The following main result of this paper is;

Theorem 5. *If the equation (1) is Briot-Bouquet type and if $\rho(0) = N \in \mathbf{N}^*$ and $\rho(x) \not\equiv \rho(0)$, then*

$$S_+ = \{U(\varphi);\ \varphi(x) \in \mathbf{C}\{x\}\}\,, \tag{7}$$

where $U(\varphi)$ is an $\widetilde{\mathcal{O}}_+$-solution of (1) having an expansion of the following form:

$$U(\varphi) = u_1^0(x)t + u_0^{e_0}(x)\phi_N(t,x) + \sum_{\substack{i+|\beta|\geq 2, |\beta|<\infty,\\ |\beta|_*\leq i+|\beta|-2}} u_i^\beta(x) t^i \Phi_N^\beta$$
$$+ w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{\substack{i+j+|\beta|\geq 2,\\ |\beta|<\infty, j\geq 1,\\ |\beta|_*\leq i+j+|\beta|-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1\\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)}\{\log t\}^k \Phi_N^\beta,$$

where $u_N^0(x) \equiv 0$, $w_{0,1,0}^0(x) = \varphi(x)$ is arbitrary holomorphic function and the other coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk, and

$$l = (l_1, \dots, l_n) \in \mathbf{N}^n,\ |l| = l_1 + \cdots + l_n,\ \beta = (\beta_l \in \mathbf{N};\ l \in \mathbf{N}^n),$$
$$|\beta| = \sum_{|l|\geq 0} \beta_l,\ |\beta|_p = \sum_{|l|=p} \beta_l \text{ for } p \geq 0,\ |\beta|_* = \sum_{|l|\geq 2} (|l|-1)\beta_l,$$
$$\Phi_N^\beta = \prod_{|l|\geq 0} \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l},\ \partial_x^l = \partial_1^{l_1} \cdots \partial_n^{l_n},\ \phi_N(t,x) = \frac{t^{\rho(x)} - t^N}{\rho(x) - N}.$$

The following lemma will play an important role in the proof of Theorem 5.

At first, we define some notations. We denote for $l \in \mathbf{N}^n$, $e_l = (\beta_k;\ k \in \mathbf{N}^n)$ with $\beta_l = 1$ and $\beta_k = 0$ for $k \neq l$ and for $p \in \mathbf{N}$, $e(p) = (i_1, \dots, i_n)$ with $i_p = 1$ and $i_q = 0$ for $q \neq p$, and denote that $l^1 < l^0$ is defined by $|l^1| < |l^0|$ and $l_i^1 \leq l_i^0$ for $i = 1, \dots, n$.

Lemma 6. *Let $\rho(x)$, ϕ_N and Φ_N^β be in Theorem 5. Then we have;*

1. $\partial_p \Phi_N^\beta = \sum_{|l|\geq 0} \beta_l (l_p + 1) \Phi_N^{\beta - e_l + e_{l+e(p)}}$ *for $i = 1, \dots, n$,*
2. $t\partial_t \phi_N = \rho(x)\phi_N + t^N$,
3. $t\partial_t \Phi_N^\beta = |\beta|\rho(x)\Phi_N^\beta + \beta_0 t^N \Phi_N^{\beta - e_0} + \sum_{|l^0|\geq 1} \sum_{l^1 < l^0} \beta_{l^0} \frac{\partial_x^{l^0 - l^1}\rho(x)}{l^0 - l^1} \Phi_N^{\beta - e_{l^0} + e_{l^1}}$.

Proof.

1. By $\partial_p (\partial_x^l \phi_N / l!)^{\beta_l} = \beta_l (\partial_x^l \phi_N / l!)^{\beta_l - 1} \partial_x^{l+e(p)} \phi_N / l!$, we have the result 1.

2. By $t\partial_t\phi_N = (\rho(x)t^{\rho(x)} - Nt^N)/(\rho(x) - N)$, we have the result 2.
3. By 2, we have

$$t\partial_t \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l} = \beta_l \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l - 1} \frac{\partial_x^l(\rho(x)\phi_N + t^N)}{l!}. \tag{8}$$

Therefore we have

$$t\partial_t \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l} = \\ = \begin{cases} \beta_0\rho(x)\phi_N^{\beta_0} + \beta_0 t^N \phi_N^{\beta_0 - 1} & \text{if } l = 0 \\ \beta_l \phi(x) \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l} + \sum_{0 \le l^1 < l} \beta_l \frac{\partial_x^{l-l^1}\rho(x)}{(l-l^1)!} \frac{\partial_x^{l^1}\phi_N}{l^1!} \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l - 1} & \text{if } |l| > 0. \end{cases}$$

Hence we have the desired result. Q.E.D.

2 Construction of formal solutions in the case $\rho(0) = 1$

By [2] (Gérard-Tahara), if the equation (1) is of Briot-Bouquet type with respect to t, then it is enough to consider the following equation:

$$Lu = t\partial_t u - \rho(x)u = a(x)t + G_2(x)(t, u, \partial_x u) \tag{9}$$

where $\rho(x)$ and $a(x)$ are holomorphic functions in a neighborhood of the origin, and the function $G_2(x)(t, X_0, X_1, \ldots, X_n)$ is a holomorphic function in a neighborhood of the origin in $\mathbf{C}_x^n \times \mathbf{C}_t \times \mathbf{C}_{X_0} \times \mathbf{C}_{X_1} \times \cdots \times \mathbf{C}_{X_n}$ with the following expansion:

$$G_2(x)(t, X_0, X_1, \ldots, X_n) = \sum_{p+|\alpha|\ge 2} a_{p,\alpha}(x)t^p\{X_0\}^{\alpha_0}\{X_1\}^{\alpha_1}\cdots\{X_n\}^{\alpha_n} \tag{10}$$

and we may assume that the coefficients $\{a_{p,\alpha}(x)\}_{p+|\alpha|\ge 2}$ are holomorphic functions on D_R for a sufficiently small $R > 0$. We put $A_{p,\alpha}(R) := \max_{x\in D_R} |a_{p,\alpha}(x)|$ for $p + |\alpha| \ge 2$. Then for $0 < r < R$

$$\sum_{p+|\alpha|\ge 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t^p X_0^{\alpha_0} X_1^{\alpha_1} \times \cdots \times X_n^{\alpha_n} \tag{11}$$

is convergent in a neighborhood of the origin.

In this section, we assume $\rho(0) = 1$ and $\rho(x) \not\equiv 1$ and we will construct formal solutions of the equation (9).

Proposition 7. *If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, the equation (9) has a family of formal solutions of the form:*

$$\begin{aligned} u = {} & u_0^{e_0}(x)\phi_1 + \sum_{m\ge 2} \sum_{\substack{i+|\beta|=m \\ |\beta|_* \le m-2}} u_i^\beta(x) t^i \Phi_1^\beta \\ & + w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{m\ge 2} \sum_{\substack{i+j+|\beta|=m \\ j\ge 1, |\beta|_*\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \end{aligned} \tag{12}$$

where $w_{0,1,0}^0(x)$ *is an arbitrary holomorphic function and the other coefficients* $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ *are holomorphic functions determined by* $w_{0,1,0}^0(x)$ *and defined in a common disk.*

Remark 8. By the relation $|\beta|_* \le m-2$ in summations of the above formal solution, we have $\beta_l = 0$ for any $l \in \mathbf{N}^n$ with $|l| \ge m$.

We define the following two sets U_m and W_m for $m \ge 1$ to prove Proposition 7.

Definition 9. We denote by U_m the set of all functions u_m of the following forms:

$$\begin{aligned} u_1 &= u_1^0(x)t + u_0^{e_0}(x)\phi_1, \\ u_m &= \sum_{\substack{i+|\beta|=m \\ |\beta|_* \le m-2}} u_i^\beta(x) t^i \Phi_1^\beta \text{ for } m \ge 2, \end{aligned} \tag{13}$$

and denote by W_m the set of all functions w_m of the following forms:

$$\begin{aligned} w_1 &= w_{0,1,0}^0(x) t^{\rho(x)}, \\ w_m &= \sum_{\substack{i+j+|\beta|=m \\ j \ge 1, |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \text{ for } m \ge 2 \end{aligned}$$

where $u_i^\beta(x)$, $w_{i,j,k}^\beta(x) \in \mathbf{C}\{x\}$.

We can rewrite the formal solution (12) as follows:

$$u = \sum_{m \ge 1} (u_m + w_m) \text{ where } u_m \in U_m,\ w_m \in W_m. \tag{14}$$

Let us show important relations of u_m and w_m for $m \ge 2$. By Lemma 6, we have

$$\begin{aligned} \partial_p u_m &= \sum_{\substack{i+|\beta|=m \\ |\beta|_* \le m-2}} \left\{ \partial_p u_i^\beta(x) t^i \Phi_1^\beta + \sum_{|l|=0}^{m-1} (l_p+1)\beta_l u_i^\beta(x) t^i \Phi_1^{\beta - e_l + e_{l+e(p)}} \right\}, \\ \partial_p w_m &= \sum_{\substack{i+j+|\beta|=m \\ j \ge 1, |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \Big\{ \partial_p w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \\ &\qquad + j\partial_p\rho(x) w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^{k+1} \Phi_1^\beta \\ &\qquad + \sum_{|l|=0}^{m-1} (l_p+1)\beta_l w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta - e_l + e_{l+e(p)}} \Big\} \end{aligned} \tag{15}$$

for $p = 1, \ldots, n$, and we have

$$Lu_m = \sum_{\substack{i+|\beta|=m \\ |\beta|_* \le m-2}} \Big\{ \{i + (|\beta| - 1)\rho(x)\} u_i^\beta(x) t^i \Phi_1^\beta + \beta_0 u_i^\beta(x) t^{i+1} \Phi_1^{\beta - e_0} + \sum_{|l^0|=1}^{m-1} \sum_{l^1 < l^0} \beta_{l^0} \frac{\partial_x^{l^0 - l^1} \rho(x)}{(l^0 - l^1)!} u_i^\beta(x) t^i \Phi_1^{\beta - e_{l^0} + e_{l^1}} \Big\}, \tag{16}$$

$$\begin{aligned} Lw_m = \sum_{\substack{i+j+|\beta|=m \\ j\ge 1, |\beta|_* \le m-2}} \sum_{\substack{k \le i + |\beta|_0 + |\beta|_1 \\ +2(j-1)}} \Big\{ &\{i + (j + |\beta| - 1)\rho(x)\} \\ &\times w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \\ + k w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^{k-1} \Phi_1^\beta &+ \beta_0 w_{i,j,k}^\beta(x) t^{i+j\rho(x)+1} \{\log t\}^k \Phi_1^{\beta - e_0} \\ + \sum_{|l^0|=1}^{m-1} \sum_{l^1 < l^0} \beta_{l^0} &\frac{\partial_x^{l^0 - l^1} \rho(x)}{(l^0 - l^1)!} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta - e_{l^0} + e_{l^1}} \Big\}. \end{aligned}$$

We show two lemma.

Lemma 10. *If $u_m \in U_m$ and $w_m \in W_m$, then $Lu_m \in U_m$ and $Lw_m \in W_m$.*

Proof. We prove $Lu_m \in U_m$. We will see all powers of each terms in (16). For the second term in (16), we have $i+1+|\beta - e_0| = i + |\beta| = m$ and $[\beta - e_0] = [\beta] \le m-2$.

For the third term, we have $i+|\beta - e_{l^0} + e_{l^1}| = i + |\beta| = m$ and $[\beta - e_{l^0} + e_{l^1}] = [\beta]$ (if $|l^0| = 1$), $= [\beta] - (|l^0| - 1)$ (if $|l^0| > 1$ and $|l^1| \le 1$), $= [\beta] - |l^0| + |l^1|$ (if $|l^0| > 1$ and $|l^1| > 1$). Further by $l^1 < l^0$, we have $[\beta - e_{l^0} + e_{l^1}] \le [\beta] \le m-2$. Hence we have $Lu_m \in U_m$.

We can prove $Lw_m \in W_m$ as $Lu_m \in U_m$, and we omit the details. Q.E.D.

Lemma 11. *If $u_m \in U_m$ and $w_m \in W_m$, then the following relations hold by the relation (15) for $i, j = 1, \ldots, n$*

1. *$a(x)U_m \subset U_m$ and $a(x)W_m \subset W_m$ for any holomorphic function $a(x)$,*
2. *tU_m, $\phi_1 U_m \subset U_{m+1}$ and $t^{\rho(x)}U_m$, tW_m, $t^{\rho(x)}W_m$, $\phi_1 W_m \subset W_{m+1}$,*
3. *$u_m \times u_n$, $\partial_i u_m \times \partial_j u_n$, $\partial_i u_m \times u_n \in U_{m+n}$,*
4. *$w_m \times w_n$, $\partial_i w_m \times \partial_j w_n$, $\partial_i w_m \times w_n$, $\in W_{m+n}$,*
5. *$u_m \times w_n$, $\partial_i u_m \times w_n$, $u_m \times \partial_j w_n$, $\partial_i u_m \times \partial_j w_n \in W_{m+n}$.*

Proof. This is verified by the relations (15) and (16) but tedious calculations. We may omit the details. Q.E.D.

Let us show that u_m and w_m are determined inductively on $m \ge 1$. By substituting $\sum_{m\ge 1}(u_m + w_m)$ into (9), we have

$$(1 - \rho(x)) u_1^0(x) + u_0^{e_0}(x) = a(x), \tag{17}$$

for $m \geq 2$

$$Lu_m = \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}}, \tag{18}$$

$$\begin{aligned} Lw_m = & \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} (u_{m_{0,h_0}} + w_{m_{0,h_0}}) \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j (u_{m_{j,h_j}} + w_{m_{j,h_j}}) \\ & - \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}}, \end{aligned} \tag{19}$$

where $|m_n| = \sum_{i=0}^{n} m_i(\alpha_i)$ and $m_i(\alpha_i) = m_{i,1} + \cdots + m_{i,\alpha_i}$ for $i = 0, 1, \ldots, n$.

We take any holomorphic function $\varphi(x) \in \mathbf{C}\{x\}$ and put $w^0_{0,1,0}(x) = \varphi(x)$, and by (17), we put $u^0_1(x) \equiv 0$ and $u^{e_0}_0(x) = a(x)$.

For $m \geq 2$, let us show that u_m and w_m are determined by induction. By Lemma 11, the right side of (18) belongs to U_m and the right side of (19) belongs to W_m. Further by $m_{j,h_j} \geq 1$, we have $m_{j,h_j} < m$ for $h_j = 1, \ldots, \alpha_j$ and $j = 0, \ldots, n$. Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi^\beta_1$ and $t^{i+j\rho(x)}\{\log t\}^k \Phi^\beta_1$ respectively for (18) and (19), then put

$$\begin{aligned} & \{i + (|\beta| - 1)\rho(x)\} u^\beta_i(x) \\ & + (\beta_0 + 1) u^{\beta+e_0}_{i-1}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0\leq l^1 < l^0} (\beta_{l^0} + 1) \frac{\partial_x^{l^0 - l^1}\rho(x)}{(l^0 - l^1)!} u_i^{\beta + e_{l^0} - e_{l^1}}(x) \\ & = f^\beta_i(\{a_{p,\alpha}\}_{2\leq p+|\alpha|\leq m},\ \{u^{\beta'}_{i'}(x)\}_{i'+|\beta'|<m}) \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \{i + (j + |\beta| - 1)\rho(x)\} w^\beta_{i,j,k}(x) + (k+1) w^\beta_{i,j,k+1}(x) \\ & + (\beta_0 + 1) w^{\beta+e_0}_{i-1,j,k}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0\leq l^1 < l^0} (\beta_{l^0} + 1) \frac{\partial_x^{l^0 - l^1}\rho(x)}{(l^0 - l^1)!} w_{i,j,k}^{\beta + e_{l^0} - e_{l^1}}(x) \\ & = g^\beta_{i,j,k}(\{a_{p,\alpha}\}_{2\leq p+|\alpha|\leq m},\ \{u^{\beta'}_{i'}(x)\}_{i'+|\beta'|<m},\ \{w^{\beta'}_{i',j',k'}(x)\}_{i'+j'+|\beta'|<m}). \end{aligned} \tag{21}$$

We define an order for the multi indices (i, β) and (i, j, k, β) to show that $u^\beta_i(x)$ and $w^\beta_{i,j,k}(x)$ are determined by (20) and (21).

Definition 12. The relation $(i', \beta') < (i, \beta)$ is defined by the following orders;
1. $i' + |\beta'| < i + |\beta|$.
2. If $i' + |\beta'| = i + |\beta|$, then $i' < i$.

3. If $i'+|\beta'|=i+|\beta|$ and $i'=i$, then $|\beta'|_0<|\beta|_0$.
4. If $i'+|\beta'|=i+|\beta|$, $i'=i$, $|\beta'|_0=|\beta|_0,\dots,|\beta'|_l=|\beta|_l$, then $|\beta'|_{l+1}<|\beta|_{l+1}$.

The relation $(i',j',k',\beta')<(i,j,k,\beta)$ is defined by the following orders;
1. $i'+j'+|\beta'|<i+j+|\beta|$.
2. If $i'+j'+|\beta'|=i+j+|\beta|$, then $i'<i$.
3. If $i'+j'+|\beta'|=i+j+|\beta|$ and $i'=i$, then $j'<j$.
4. If $i'+j'+|\beta'|=i+j+|\beta|$, $i'=i$ and $j'=j$, then $|\beta'|_0<|\beta|_0$.
5. If $i'+j'+|\beta'|=i+j+|\beta|$, $i'=i$, $j'=j$, $|\beta'|_0=|\beta|_0,\dots,|\beta'|_l=|\beta|_l$, then $|\beta'|_{l+1}<|\beta|_{l+1}$.
6. If $(i',j',\beta')=(i,j,\beta)$, then $k'>k$.

For $m\ge 2$, we have $i+(|\beta|-1)\rho(x)\neq 0$ and $i+(j+|\beta|-1)\rho(x)\neq 0$ by $\rho(0)=1$. Therefore all the coefficients $u_i^\beta(x)$ and $w_{i,j,k}^\beta(x)$ are determined in the order of Definition 12. Hence we obtain Proposition 7. Q.E.D.

3 Convergence of the formal solutions in the case $\rho(0)=1$

In this section, we show that the formal solution (12) converges in $\widetilde{\mathcal{O}}_+$.

Proposition 13. *Let γ satisfy $0<\gamma<1$ and let λ be sufficiently large. Then for any sufficiently small $r>0$ we have the following result;*

For any $\theta>0$ there is an $\epsilon>0$ such that the formal solution (12) converges in the following region:

$$\begin{aligned}\{(t,x)\in \mathbf{C}_t\times\mathbf{C}_x^n;\quad &|\eta(t,\lambda)t|<\epsilon,\ |\eta(t,\lambda)^2t^{\rho(x)}|<\epsilon,\\ &|\eta(t,\lambda)t^\gamma|<\epsilon,\ \ t\in S_\theta \text{ and } x\in D_r\},\end{aligned}$$

where $\eta(t,\lambda)=\max\{|(\log t)/\lambda|,\ 1\}$.

In this section, we put $w_{i,0,0}^\beta(x):=u_i^\beta(x)$ and $w_{i,0,k}^\beta(x)\equiv 0$ for $k\ge 1$ in the formal solution (12). Then the formal solution (12) is as follows:

$$\begin{aligned}u=\ &w_{0,0,0}^{e_0}(x)\phi_1+w_{0,1,0}^0(x)t^{\rho(x)}\\ &+\sum_{m\ge 2}\sum_{\substack{i+j+|\beta|=m\\|\beta|_*\le m-2}}\sum_{\substack{k\le i+|\beta|_0+|\beta|_1\\+2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^\beta.\end{aligned}\tag{22}$$

Let us define the following set V_m for (22).

Definition 14. We denote by V_m the set of all the functions v_m of the following forms:

$$\begin{aligned}v_1&=w_{0,0,0}^{e_0}(x)\phi_1+w_{0,1,0}^0(x)t^{\rho(x)},\\ v_m&=\sum_{\substack{i+j+|\beta|=m\\|\beta|_*\le m-2}}\sum_{\substack{k\le i+|\beta|_0+|\beta|_1\\+2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^\beta\quad\text{for}\quad m\ge 2.\end{aligned}\tag{23}$$

We define the following estimate for the function v_m.

Definition 15. For the function (23), we define

$$
\begin{aligned}
||v_1||_{r,c,\lambda} &= ||v_1||_{r,c} := \frac{||w^{e_0}_{0,0,0}||_r}{c} + ||w^{0}_{0,1,0}||_r, \\
||v_m||_{r,c,\lambda} &:= \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \le m-2}} \ \sum_{\substack{k \le i+|\beta|_0+\beta_1 \\ +2(j-1)}} \frac{||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta>}} \quad \text{for} \quad m \ge 2
\end{aligned}
\tag{24}
$$

for $c > 0$ and $\lambda > 0$, where

$$
||w^{\beta}_{i,j,k}||_r = \max_{x \in D_r} |w^{\beta}_{i,j,k}(x)| \text{ and } <\beta> = \sum_{|l| \ge 0} (|l|+1)\beta_l. \tag{25}
$$

We will make use of

Lemma 16. *For a holomorphic function $f(x)$ on D_R, we have*

$$
||\partial^{\alpha}_x f||_{R_0} \le \frac{\alpha!}{(R-R_0)^{|\alpha|}} ||f||_R \quad \textit{for} \quad 0 < R_0 < R. \tag{26}
$$

Proof. By Cauchy's integral formula, we have the desired result, and we omit the details. Q.E.D

Lemma 17. *If a holomorphic function $f(x)$ on D_R satisfies*

$$
||f||_{R_0} \le \frac{C}{(R-r)^p} \quad \textit{for} \quad 0 < r < R \tag{27}
$$

then we have

$$
||\partial_i f||_{R_0} \le \frac{Ce(p+1)}{(R-r)^{p+1}} \quad \textit{for} \quad 0 < r < R, \quad i = 1, \ldots, n. \tag{28}
$$

For the proof, see Hörmander ([5]lemma 5.1.3)

Let us show the following estimate for the function Lv_m.

Lemma 18. *Let $0 < R_0 < R$. Then there exists a positive constant σ such that for $m \ge 2$, if $v_m \in V_m$ we have*

$$
||Lv_m||_{r,c,\lambda} \ge \frac{\sigma}{2} m ||v_m||_{r,c,\lambda} \quad \textit{for} \quad 0 < r \le R_0 \tag{29}
$$

for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$.

Proof. Let us give an estimate the second, the third and the fourth term in the right side of the second relation in (16) respectively.

For the second term, since $k \le i+|\beta|_0+|\beta|_1+2(j-1) \le 2m$ by $i+j+|\beta|=m$ we have

$$T_2 := \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} k \frac{||w^{\beta}_{i,j,k+1}||_r \lambda^{k-1}}{c^{<\beta>}} \le \frac{2m}{\lambda} ||v_m||_{r,c,\lambda}. \tag{30}$$

For the fourth term, we have

$$\begin{aligned} T_4 := & \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \sum_{|l^0|=1}^{m-1} \sum_{l^1<l^0} \frac{\beta_{l^0}}{(l^0-l^1)!} \frac{||\partial_x^{l^0-l^1} \rho w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta-e_{l^0}+e_{l^1}>}} \\ \le & \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \sum_{|l^0|=1}^{m-1} \sum_{l^1<l^0} c^{|l^0|-|l^1|} \beta_{l^0} \frac{||\partial_x^{l^0-l^1} \rho||_{R_0}}{(l^0-l^1)!} \frac{||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta>}}. \end{aligned} \tag{31}$$

By Lemma 16, we have

$$\begin{aligned} \sum_{l^1<l^0} c^{|l^0|-|l^1|} \frac{||\partial_x^{l^0-l^1} \rho||_{R_0}}{(l^0-l^1)!} & \le \sum_{l^1<l^0} \left(\frac{c}{R-R_0} \right)^{|l^0|-|l^1|} ||\rho||_R \\ & \le \frac{cn||\rho||_R}{R-R_0} \left(\frac{R-R_0}{R-R_0-c} \right)^n \end{aligned} \tag{32}$$

for sufficiently small $c>0$. Therefore by (31) and (32), we have

$$T_4 \le \kappa(c) \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \sum_{|l^0|=1}^{m-1} \beta_{l^0} \frac{||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta>}} \tag{33}$$

where $\kappa(c) := \frac{cn}{R-R_0} (\frac{R-R_0}{R-R_0-c})^n ||\rho||_R$.

For the third term, we have

$$\begin{aligned} T_3 : = & \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \beta_0 \frac{||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta-e_0>}} \\ = & \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \le m-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} c\beta_0 \frac{||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta>}}. \end{aligned}$$

Therefore, since $c\beta_0 + \kappa(c)\sum_{|l^0|=1}^{m-1}\beta_{l^0} \leq \frac{\sigma}{3}m$ by the conditions $\kappa(0) = 0$ and $i + j + |\beta| = m \geq 2$ for sufficiently small $c > 0$ and some $\sigma > 0$ we have

$$T_2 + T_3 + T_4 \leq \left(\frac{2m}{\lambda} + \frac{\sigma}{3}m\right) ||v_m||_{r,c,\lambda}. \tag{34}$$

Further we have $|i + (j + |\beta| - 1)\rho(x)| \geq \sigma m$ by the condition $\rho(0) = 1$ and $i + j + |\beta| = m \geq 2$. Therefore we have

$$||Lv_m||_{r,c\lambda} \geq \left(\sigma m - \frac{2m}{\lambda} - \frac{\sigma}{3}m\right) ||v_m||_{r,c,\lambda}. \tag{35}$$

Hence for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$, we obtain the desired result. Q.E.D.

Let us estimate the function $\partial_i v_m$.

Definition 19. For the function $v_m \in V_m$ we define

$$D_p v_m := \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \partial_p w^{\beta}_{i,j,k}(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta} \tag{36}$$

for $p = 1, \dots, n$.

Lemma 20. *If $v_m \in V_m$, then for $i = 1, \dots, n$, we have*

$$||\partial_i v_m||_{r,c,\lambda} \leq ||D_i v_m||_{r,c,\lambda} + c_0\lambda m ||v_m||_{r,c,\lambda} + \frac{3m-2}{c}||v_m||_{r,c,\lambda} \quad \text{for} \quad 0 < r \leq R_0. \tag{37}$$

Proof. We have

$$\sum_{|l|\geq 0} (l_p + 1)\beta_l \leq \sum_{|l|=0}^{m-1} (|l| + 1)\beta_l = 2|\beta| + [\beta] \leq 3m - 2. \tag{38}$$

We put $c_0 = \max_{i=1,\dots,n}\{||\partial_i\rho||_{R_0}\}$, and by the relations (15), (38) and $j \leq m$ we obtain the desired estimate. Q.E.D.

Therefore by the relations (18), (19) and Lemma 18, 20, we have the following lemma.

Lemma 21. *If $u = \sum_{m\geq 1} v_m$ is a formal solution of the equation (9) constructing in Section 2, we have the following inequality for v_m $(m \geq 2)$:*

$$\begin{aligned} &||Lv_m||_{r,c,\lambda} \\ &\leq \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} ||a_{p,\alpha}||_r \prod_{h_0=1}^{\alpha_0} ||v_{m_{0,h_0}}||_{r,c,\lambda} \\ &\times \prod_{i=1}^{n}\prod_{h_i=1}^{\alpha_i} \{||D_i v_{m_{i,h_i}}||_{r,c,\lambda} + c_0\lambda m_{i,h_i}||v_{m_{i,h_i}}||_{r,c,\lambda} + \frac{3m_{i,h_i}-2}{c}||v_{m_{i,h_i}}||_{r,c,\lambda}\}. \end{aligned}$$

Let us define a majorant equation to show that the formal solution (22) converges.
We take A_1 so that

$$\frac{||w^{e_0}_{0,0,0}||_R}{c} + ||w^0_{0,1,0}||_R \leq A_1,$$
$$\frac{||\partial_i w^{e_0}_{0,0,0}||_R}{c} + ||\partial_i w^0_{0,1,0}||_R \leq A_1$$

for $i = 1, \dots, n$.

Then we consider the following equation:

$$\begin{aligned} \frac{\sigma}{2} Y = & \frac{\sigma}{2} A_1 t_1 \\ & + \frac{1}{R-r} \sum_{p+|\alpha|\geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} {t_1}^p Y^{\alpha_0} \prod_{i=1}^{n} \left(eY + c_0 \lambda Y + \frac{3}{c} Y \right)^{\alpha_i}. \end{aligned} \tag{39}$$

The equation (39) has a unique holomorphic solution $Y = Y(t_1)$ with $Y(0) = 0$ at $(Y, t_1) = (0, 0)$ by implicit function theorem. By an easy calculation, the solution $Y = Y(t_1)$ has the following form:

$$Y = \sum_{m \geq 1} Y_m {t_1}^m \text{ with } Y_m = \frac{C_m}{(R-r)^{m-1}} \tag{40}$$

where $Y_1 = C_1 = A_1$ and $C_m \geq 0$ for $m \geq 1$.
Then we have;

Lemma 22. *For $m \geq 1$, we have*

$$m||v_m||_{r,c,\lambda} \leq Y_m \quad \textit{for} \quad 0 < r \leq R_0 \tag{41}$$

$$||D_i v_m||_{r,c,\lambda} \leq e Y_m \quad \textit{for} \quad 0 < r \leq R_0, \tag{42}$$

for $i = 1, \dots, n$.

Proof. By $A_1 = Y_1$ and the definition of A_1, (41) and (42) hold for $m = 1$.

By induction on m, let us show that (41) and (42) hold for $m \geq 2$. By substituting the solution $Y = \sum_{m\geq 1} Y_m {t_1}^m$ into the equation (39), we have the following relation:

$$\begin{aligned} \frac{\sigma}{2} Y_m = & \frac{1}{R-r} \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} \prod_{h_0=1}^{\alpha_0} Y_{m_{0,h_0}} \\ & \times \prod_{i=1}^{n} \prod_{h_i=1}^{\alpha_i} \left\{ e Y_{m_{i,h_i}} + c_0 \lambda Y_{m_{i,h_i}} + \frac{3}{c} Y_{m_{i,h_i}} \right\} \end{aligned} \tag{43}$$

for $m \geq 2$. Therefore if we assume that (41) and (42) hold for $m_{i,h_i} < m$, by (43), Lemma 18 and Lemma 21 we obtain

$$\frac{\sigma}{2} m||v_m||_{r,c,\lambda} \leq (R-r)\frac{\sigma}{2} Y_m. \tag{44}$$

Therefore we have

$$m||v_m||_{r,c,\lambda} \leq (R-r)Y_m \leq Y_m. \tag{45}$$

The relation (45) is rewrited as follows:

$$m \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta>}} \leq \frac{C_m}{(R-r)^{m-2}}. \tag{46}$$

By (46) and Lemma 17, we have

$$m||D_i v_m||_{r,c,\lambda} \leq \frac{(m-1)eC_m}{(R-r)^{m-1}} \tag{47}$$

for $i = 1, \ldots, n$ and $0 < r < R < 1$. Therefore we have

$$||D_i v_m||_{r,c,\lambda} \leq \frac{eC_m}{(R-r)^{m-1}} = eY_m. \tag{48}$$

Hence (41) and (42) hold for $m \geq 2$. Q.E.D.

Let us show that the formal solution (22) converges by using (41) in Lemma 22. We put (22) as follows:

$$\begin{aligned} u &= u_0^{e_0}(x)\phi_1 + w^0_{0,1,0}(x)t^{\rho(x)} \\ &\quad + \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ |\beta|_* \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{w^{\beta}_{i,j,k}(x)\lambda^k}{c^{<\beta>}} t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right)^k \Psi^{\beta}_1, \end{aligned}$$

where

$$\Psi^{\beta}_1 = \prod_{|l| \geq 0} \left(c^{|l|+1} \frac{\partial^l_x \phi_1}{l!} \right)^{\beta_l}. \tag{49}$$

Firstly let us estimate (49). For $||\phi_1||_R$, we have the following lemma.

Lemma 23. *For any γ with $0 < \gamma < 1$, there is an $R > 0$ such that*

$$||\phi_1||_R = O\left(|t|^{\gamma}\right) \ \text{as } t \to 0 \text{ in } S_\theta \tag{50}$$

holds for any $\theta > 0$.

Proof. We put

$$\phi_1 = t^{\gamma}\frac{t^{\rho_0(x)+\alpha}-t^{\alpha}}{\rho_0(x)} \tag{51}$$

with $\alpha+\gamma=1$ and $\rho_0(x)=\rho(x)-1$. Then we can take $R>0$ with

$$||\rho_0||_R < \alpha \tag{52}$$

by $\rho_0(0)=0$. Therefore we have

$$\left|\left|\frac{t^{\rho_0(x)+\alpha}-t^{\alpha}}{\rho_0(x)}\right|\right|_R \leq |\log t||t|^{\alpha-||\rho_0||_R} \to 0 \quad \text{as} \quad t\to 0 \quad \text{in} \quad S_\theta \tag{53}$$

for and any $\theta>0$. Hence we have the desired result. Q.E.D.

By Lemma 23, there exists a positive constant c_1 such that

$$||\phi_1||_R \leq c_1|t|^{\gamma} \quad \text{in} \quad S_\theta. \tag{54}$$

By Lemma 16 and (54), for $|l|\geq 0$ we have

$$||\partial_x^l\phi_1||_{R_0} \leq \frac{l!}{(R-R_0)^{|l|}}||\phi_1||_R \leq \frac{l!c_1}{(R-R_0)^{|l|}}|t|^{\gamma} \quad \text{for} \quad 0<R_0<R. \tag{55}$$

Therefore, we have

$$||\Psi_1^{\beta}||_{R_0} \leq \prod_{|l|\geq 0}\left(c^{|l|+1}\frac{c_1}{(R-R_0)^{|l|}}|t|^{\gamma}\right)^{\beta_l} \leq \left(\frac{c}{R-R_0}\right)^{<\beta>}(c_1(R-R_0)|t|^{\gamma})^{|\beta|} \tag{56}$$

for $0<R_0<R$ in S_θ.

Let us estimate $t^{i+j\rho(x)}\left(\dfrac{\log t}{\lambda}\right)^k \Psi_1^{\beta}$.

We put $\eta(t,\lambda)=\max\left\{\left|\frac{\log t}{\lambda}\right|,\ 1\right\}$, $c_2=\max\left\{\frac{c}{R-R_0},1\right\}$ and $c_3=c_1(R-R_0)$. Since we have

$$<\beta> \leq 2|\beta|+|\beta|_* \leq i+j+3|\beta| \tag{57}$$

and

$$k \leq i+|\beta|_0+|\beta|_1+2(j-1) \leq i+|\beta|+2j, \tag{58}$$

we obtain

$$\left|\left|t^{i+j\rho(x)}\left(\frac{\log t}{\lambda}\right)^k\Psi_1^{\beta}\right|\right|_r \leq$$

$$\leq \{|c_2\eta(t,\lambda)t|\}^i\left\{||c_2\eta(t,\lambda)^2t^{\rho(x)}||_r\right\}^j\left\{|(c_2)^3c_3\eta(t,\lambda)t^{\gamma}|\right\}^{|\beta|}$$

in S_θ. For any sufficiently small $\epsilon>0$, there exists a sufficiently small $|t|$ in S_θ such that

$$|c_2\eta(t,\lambda)t|<\epsilon,\ \ ||c_2\eta(t,\lambda)^2t^{\rho(x)}||_r<\epsilon,\ \ |(c_2)^3c_3\eta(t,\lambda)t^{\gamma}|<\epsilon, \tag{59}$$

and we obtain

$$\left|\left|t^{i+j\rho(x)}\left(\frac{\log t}{\lambda}\right)\Psi_1^\beta\right|\right|_r \le \epsilon^m. \tag{60}$$

Then by Lemma 22, we have

$$||u||_r \le \sum_{m\ge 1} Y_m \epsilon^m \tag{61}$$

for sufficiently small $|t|$ in S_θ. Hence the formal solution (22) converges for $x \in D_r$ and sufficiently small $|t|$ in S_θ. Q.E.D.

4 Completion of the proof of Theorem 5 in the case $\rho(0) = 1$

In this section, let us complete the proof of Theorem 5 in the case $\rho(0) = 1$.

We know the following theorem.

Theorem 24. *If $u_i(t,x) \in \widetilde{\mathcal{O}}_+$ $(i = 1, 2)$ are solutions of (9), we have;*

1. For any $a < \rho(0) = 1$, we have $t^{-a}(u_1 - u_2) \in \widetilde{\mathcal{O}}_+$.

2. If $t^{-b}(u_1 - u_2) \in \widetilde{\mathcal{O}}_+$ for some $b \ge \rho(0) = 1$, we have $u_1(t,x) = u_2(t,x)$ in $\widetilde{\mathcal{O}}_+$.

For the proof, see Gérard and Tahara ([2] Theorem 3).

By the discussions in sections 2, 3 and 4, we already know the following results;

(C1) If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, for any $\varphi(x) \in \mathbf{C}\{x\}$, the equation (1) has a unique $\widetilde{\mathcal{O}}_+$-solution $U(\varphi)(t,x)$ having an expansion of the form

$$\begin{aligned} U(\varphi) = {} & w_{0,0,0}^{e_0}(x)\phi_1 + w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{m\ge 2}\sum_{\substack{i+|\beta|=m \\ |\beta|_*\le m-2}} u_i^\beta(x)t^i \Phi_1^\beta \\ & + \sum_{m\ge 2}\sum_{\substack{i+j+|\beta|=m \\ j\ge 1, |\beta|_*\le m-2}}\sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)}\{\log t\}^k \Phi_1^\beta \end{aligned} \tag{62}$$

with $w_{0,1,0}^0(x) = \varphi(x)$, where all the coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic in a common disk centered at the origin of $\mathbf{C}_x^n$. If we take $\varphi(x) = 0$, then the solution $u_0(t,x)$ has the expansion

$$u_0(t,x) = U(0) = u_0^{e_0}(x)\phi_1 + \sum_{m\ge 2}\sum_{\substack{i+|\beta|=m \\ |\beta|_*\le m-2}} u_i^\beta(x)t^i\Phi_1^\beta. \tag{63}$$

(C2) If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, and if a solution $u(t,x) \in \widetilde{\mathcal{O}}_+$ of the equation (1) is expressed in the form

$$t^{-1}\left(u(t,x) - u_0^{e_0}(x)\phi_1(t,x) - \varphi(x)t^{\rho(x)}\right) \in \widetilde{\mathcal{O}}_+, \tag{64}$$

then the coefficient $u_0^{e_0}(x)$ is uniquely determined by the equation (1), and they are independent of $\varphi(x)$.

If $\rho(0)=1$ and $\rho(x)\not\equiv 1$, by (C1) we have

$$S_+ \supset \{U(\varphi);\ \varphi(x)\in \mathbf{C}\{x\}\}\,. \tag{65}$$

Hence it is sufficient to prove the following proposition to complete the proof of the main theorem.

Proposition 25. *Assume* (A1), (A2) *and* (A3). *Let* $u_0(t,x)$ *and* $U(\varphi)(t,x)$ *be as above. If* $\rho(0)=1$ *and* $\rho(x)\not\equiv 1$, *and if* $u(t,x)\in S_+$, *then we can find a* $\varphi(x)\in\mathbf{C}\{x\}$ *such that* $u(t,x)\equiv U(\varphi)(t,x)$ *holds in* $\widetilde{\mathcal{O}}_+$.

The proof of this proposition is almost the same as that of Proposition 2 in Gérard and Tahara [1]; so we may omit the details. Q.E.D.

By (65) and Proposition 25 we obtain the main theorem 5 in the case $\rho(0)=1$ and $\rho(x)\not\equiv 1$. Q.E.D.

5 Proof of Theorem 5 in the case $\rho(0)=N$

In Section 2, 3, and 4, we have proved Theorem 5 in the case $\rho(0)=1$. In this section, we will prove Theorem 5 in the case $\rho(0)=N\geq 2$ and $\rho(x)\not\equiv N$.

We put

$$u(t,x)=\sum_{i=1}^{N-1}u_i(x)t^i+t^{N-1}w(t,x), \tag{66}$$

where $u_i(x)\in\mathbf{C}\{x\}$ $(1\leq i\leq N-1)$ and $w(t,x)\in\widetilde{\mathcal{O}}_+$.

Then by an easy calculation we see

Lemma 26. *If the function* (66) *is a solution of the equation* (9), *the functions* $u_1(x),\ldots,u_{N-1}(x)$ *are uniquely determined and* $w(t,x)$ *satisfies an equation of the following form:*

$$\begin{aligned}(t\partial_t-\rho(x)+N-1)w &= ta(t,x)+tA_0(t,x)w+t\sum_{i=1}^{n}A_i(t,x)\partial_i w \\ &\quad+\sum_{|\alpha|\geq 2}t^{(N-1)(|\alpha|-1)}A_\alpha(t,x)w^{\alpha_0}\prod_{i=1}^{n}(\partial_i w)^{\alpha_i},\end{aligned} \tag{67}$$

where

$$a(t,x)=\frac{1}{t^N}\left(G_2(x)(t,w_0,\partial_x w_0)+ta(x)-(t\partial_t-\rho(x))w_0\right) \tag{68}$$

with $w_0 = \sum_{i=1}^{N-1} u_i(x)t^i$ *and*

$$\begin{aligned} A_i(t,x) &= \frac{1}{t}\frac{\partial G_2}{\partial X_i}(x)(t, w_0, \partial_x w_0), \quad i = 0, 1, \ldots, n, \\ A_\alpha(t,x) &= \frac{1}{\alpha!}\frac{\partial^{|\alpha|} G_2}{\partial X^\alpha}(x)(t, w_0, \partial_x w_0), \quad |\alpha| \geq 2. \end{aligned}$$

Since the equation (67) satisfies the conditions (A1), (A2), (A3) and the characteristic exponents $\rho^N(x) = \rho(x) - N + 1$ satisfies $\rho^N(0) = 1$, we can apply the results in sections 2, 3 and 4.

Further, by the form of all the nonlinear parts of the equation (67), we see that the formal solution constructed in Section 2 has the following form:

$$\begin{aligned} w = {} & u_0^{N,e_0}(x)\phi_{N,1} + w_{0,1,0}^{N,0}(x)t^{\rho^N(x)} \\ & + \sum_{i \geq 2} u_i^N(x)t^i + \sum_{m \geq 2} \sum_{\substack{i+|\beta|=m \\ |\beta|_* \leq m-2, |\beta| \geq 1}} u_i^{N,\beta}(x)t^{i+(N-1)(|\beta|-1)}\Phi_{N,1}^\beta \\ & + \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ j \geq 1, |\beta|_* \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^{N,\beta}(x)t^{i+(N-1)(j+|\beta|-1)+j\rho^N(x)}\{\log t\}^k \Phi_{N,1}^\beta \end{aligned} \tag{69}$$

where $\Phi_{N,1}^\beta = \prod_{|l| \geq 0} \left(\frac{\partial_x^l \phi_{N,1}}{l!}\right)^{\beta_l}$ and $\phi_{N,1} = \dfrac{t^{\rho^N(x)} - t}{\rho^N(x) - 1}$. Therefore we have

$$\begin{aligned} u = {} & \sum_{i=1}^{N-1} u_i(x)t^i + u_0^{N,e_0}(x)\phi_N + w_{0,1,0}^{N,0}(x)t^{\rho(x)} \\ & + \sum_{i \geq 2} u_i^N(x)t^{i+N-1} + \sum_{m \geq 2} \sum_{\substack{i+|\beta|=m \\ |\beta|_* \leq m-2, |\beta| \geq 1}} u_i^{N,\beta}(x)t^i \Phi_N^\beta \\ & + \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ j \geq 1, |\beta|_* \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^{N,\beta}(x)t^{i+j\rho(x)}\{\log t\}^k \Phi_N^\beta. \end{aligned} \tag{70}$$

We put

$$\begin{aligned} u_i^N(x) \mapsto u_{i+N-1}(x) \quad \text{for} \quad i \geq 2, \quad u_i^{N,\beta}(x) \mapsto u_i^\beta(x) \quad \text{for} \quad |\beta| \geq 1, \\ w_{i,j,k}^{N,\beta}(x) \mapsto w_{i,j,k}^\beta(x) \quad \text{for any} \quad (i,j,k,\beta), \end{aligned}$$

and we have $u_N^0(x) \equiv 0$ by the form of the solution (69) and the above relations. Hence this completes the proof of Theorem 5. Q.E.D.

References

1. Briot, Ch. and Bouquet, J. Cl., *Recherches sur les propriétés des fonctions définies par des équations différentielles*, J. Ecole Polytech., 21(1856), 133–197.
2. Gérard, R. and Tahara, H., *Holomorphic and Singular Solutions of Nonlinear Singular First Order Partial Differential Equations*, Publ. RIMS, Kyoto Univ., 26(1990), 979–1000.
3. Gérard, R. and Tahara, H., *Singular Nonlinear Partial Differential Equations*, Vieweg, 1996.
4. Hill, E., *Ordinary differential equations in the complex domain*, John Wiley and Sons, 1976.
5. Hörmander, L., *Linear partial differential operators*, Springer, 1963.
6. Kimura T., *Ordinary differential equations*, Iwanami Shoten, 1977 (in Japanese).
7. Yamane, H., *Nonlinear Singular First Order Partial Differential Equations Whose Characteristic Exponent Takes a Positive Integral Value*, Publ. RIMS, Kyoto Univ., 33(1997), 801–811.