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Landesman–Lazer Type Conditions and Quasilinear Elliptic Equations

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Abstract. We study the existence of the weak solutions of nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + g(u) - h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $p > 1$, $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous function, $h \in L^{p'}(\Omega)$ ($p' = \frac{p}{p-1}$), Δ_p is the p -Laplacian, i.e. $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ and $\lambda \in \mathbb{R}$.

Our sufficient conditions generalize all previously published results.

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1 Introduction. The variational eigenvalues.

We study the existence of the weak solutions of nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + g(u) - h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

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where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $p > 1$, $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous function, $h \in L^{p'}(\Omega)$ ($p' = \frac{p}{p-1}$), $\lambda \in \mathbb{R}$ and Δ_p is the p -Laplacian, i.e.

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

We recall that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1) if and only if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v \, dx = \lambda \int_{\Omega} |u|^{p-2} uv \, dx + \int_{\Omega} g(u)v \, dx - \int_{\Omega} hv \, dx$$

for all $v \in W_0^{1,p}(\Omega)$.

It is possible to achieve that the weak solutions of our BVP (1) corresponding with the critical points of the functional

$$J(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\Omega} |u|^p \, dx - \int_{\Omega} G(u) \, dx + \int_{\Omega} hu \, dx : W_0^{1,p}(\Omega) \rightarrow \mathbb{R},$$

where

$$G(t) := \int_0^t g(s) \, ds.$$

Now we are going to investigate how the choice of λ , g and h (and their relation) influence the *geometry* of our functional J . The great part in that has the information, if λ is the eigenvalue of the operator $-\Delta_p$ or not; i.e. if there exists a weak nontrivial solution of BVP

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now we define the even functional

$$I(u) := \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx} : W_0^{1,p}(\Omega) \setminus \{0\} \rightarrow \mathbb{R},$$

and for any $k \in \mathbb{N}$ we consider set

$$\mathcal{F}_k := \left\{ \mathcal{A} \subset \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\} : \right. \\ \left. \text{there exists a continuous odd surjection } h : S^k \rightarrow \mathcal{A} \right\},$$

where S^k represents the unit sphere in \mathbb{R}^k .

Pavel Drábek and Stephen B. Robinson proved in 1999 that for any $k \in \mathbb{N}$ the number

$$\lambda_k := \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} I(u)$$

is an eigenvalue of $-\Delta_p$. This situation is very interesting, because it is not known if this represents a complete list of eigenvalues¹ but it is known that:

¹ Nobody knows how to obtain all eigenvalues of $-\Delta_p$; we only know that we have complete list of eigenvalues if $N = 1$ or $p = 2$.

- λ_1 is the first eigenvalue,

$$\lambda_1 = \min\left\{\int_{\Omega} |\nabla u|^p dx; u \in W_0^{1,p}(\Omega), \int_{\Omega} |u|^p dx = 1\right\},$$

there exists a unique positive corresponding eigenfunction φ_1 whose norm in $W_0^{1,p}(\Omega)$ is 1,

- λ_2 is the second eigenvalue,
- $\forall k \in \mathbb{N} \setminus \{1, 2\} : 0 < \lambda_1 < \lambda_2 \leq \lambda_k \leq \lambda_{k+1}$,
- $\lambda_k \rightarrow +\infty$.

Pavel Drábek and Stephen B. Robinson assumed in their paper that function g is bounded and they found some sufficient conditions for solvability of our BVP (1). Now we are going to generalize these results for some not bounded function g .

2 The case $\lambda < \lambda_1$.

Theorem 1. *If we suppose in addition that*

$$\lim_{x \rightarrow \pm\infty} \frac{g(x)}{|x|^{p-1}} = 0,$$

then the BVP (1) has at least one weak solution.

(It follows from Ekeland variational principle (see [6] and [1]) that the energy functional J has a global minimum in this case.)

3 The case $\lambda = \lambda_1$.

Theorem 2. *Let us define*

$$F(x) := \frac{p}{x} \int_0^x g(s) ds - g(x).$$

We suppose

$$\lim_{x \rightarrow \pm\infty} \frac{g(x)}{|x|^{p-1}} = 0$$

and

$$\overline{F(-\infty)} \int_{\Omega} \varphi_1(x) dx < (p-1) \int_{\Omega} h(x) \varphi_1(x) dx < \underline{F(+\infty)} \int_{\Omega} \varphi_1(x) dx, \quad (2)$$

or

$$\boxed{\overline{F(+\infty)} \int_{\Omega} \varphi_1(x) dx < (p-1) \int_{\Omega} h(x) \varphi_1(x) dx < \underline{F(-\infty)} \int_{\Omega} \varphi_1(x) dx,} \quad (3)$$

where

$$\begin{aligned} \overline{F(-\infty)} &= \limsup_{x \rightarrow -\infty} F(x), & \underline{F(+\infty)} &= \liminf_{x \rightarrow +\infty} F(x), \\ \underline{F(+\infty)} &= \limsup_{x \rightarrow +\infty} F(x), & \overline{F(-\infty)} &= \liminf_{x \rightarrow -\infty} F(x). \end{aligned}$$

Then the BVP (1) has at least one weak solution.

(If (2) is satisfied that the energy functional J has a saddle point geometry while if (3) holds this functional attains its global minimum ... see [3].)

4 The case $\lambda_k < \lambda < \lambda_{k+1}$.

Theorem 3. We suppose

$$\boxed{\lim_{x \rightarrow \pm\infty} \frac{g(x)}{|x|^{p-1}} = 0}$$

and

$$\begin{aligned} &\forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\} : \\ &(p-1) \int_{\Omega} h(x)v(x) dx < \underline{F(+\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) > 0\}}} v(x) dx + \overline{F(-\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) < 0\}}} v(x) dx, \end{aligned} \quad (4)$$

or

$$\begin{aligned} &\forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\} : \\ &(p-1) \int_{\Omega} h(x)v(x) dx > \overline{F(+\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) > 0\}}} v(x) dx + \underline{F(-\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) < 0\}}} v(x) dx, \end{aligned} \quad (5)$$

and that

$$\begin{aligned} &\forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\}, \|v\| = 1 : \\ &(\forall \delta \in \mathbb{R}^+) (\exists \eta(\delta) \in \mathbb{R}^+) : \text{meas}\{x \in \Omega : |v(x)| \leq \eta(\delta)\} < \delta \\ &(\text{"the strong unique continuation property"}). \end{aligned} \quad (6)$$

Then the BVP (1) has at least one weak solution. ²

² Notice that if $\lambda \in \mathbb{R}$ is not an eigenvalue of the $-\Delta_p$, i.e. if does not exist function $v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\}$, then the conditions (4), (5) and (6) are vacuously true.

(Proof of this theorem is based on application a saddle point theorem for linked sets ... see [1].)

5 The case $\lambda = \lambda_k$.

Theorem 4 ([1]). *We suppose*

$$\lim_{x \rightarrow \pm\infty} \frac{g(x)}{|x|^{p-1}} = 0$$

and

$$\begin{aligned} & \forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\} : \\ & (p-1) \int_{\Omega} h(x)v(x) dx < \overline{F(+\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) > 0\}}} v(x) dx + \overline{F(-\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) < 0\}}} v(x) dx, \end{aligned} \quad (7)$$

or

$$\begin{aligned} & \forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\} : \\ & (p-1) \int_{\Omega} h(x)v(x) dx > \overline{F(+\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) > 0\}}} v(x) dx + \overline{F(-\infty)} \int_{\substack{\{x \in \Omega: \\ v(x) < 0\}}} v(x) dx, \end{aligned} \quad (8)$$

and that

$$\begin{aligned} & \forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\}, \|v\| = 1 : \\ & (\forall \delta \in \mathbb{R}^+) (\exists \eta(\delta) \in \mathbb{R}^+) : \text{meas}\{x \in \Omega : |v(x)| \leq \eta(\delta)\} < \delta. \end{aligned}$$

Further, we assume that there exists sequence

$$\mu_n \searrow \lambda_k \text{ (if we assume (7))} \quad \text{or} \quad \mu_n \nearrow \lambda_k \text{ (if we assume (8))}$$

such that

$$\begin{aligned} & \forall n \in \mathbb{N} \forall v \in \text{Ker}(-\Delta_p - \mu_n) \setminus \{0\}, \|v\| = 1 : \\ & (\forall \delta \in \mathbb{R}^+) (\exists \eta(\delta) \in \mathbb{R}^+) : \text{meas}\{x \in \Omega : |v(x)| \leq \eta(\delta)\} < \delta. \end{aligned}$$

Then the BVP (1) has at least one weak solution.

6 The one dimensional case.

At the finish we note: if we consider one dimensional problem and – for example – $\Omega = (0, \pi)$, i.e. if we consider BVP

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u + g(u) - h(x) \text{ in } (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases} \quad (9)$$

then situation is easier: we know all eigenvalues of the p -Laplacian³ and we know that any eigenfunction satisfies "the strong unique continuation property". Therefore we can rewrite our results in this form:

Theorem 5. *We suppose*

$$\lim_{x \rightarrow \pm\infty} \frac{g(x)}{|x|^{p-1}} = 0$$

and

$$\begin{aligned} & \forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\} : \\ & (p-1) \int_0^\pi h(x)v(x) dx < \underline{F(+\infty)} \int_0^\pi v^+(x) dx + \overline{F(-\infty)} \int_0^\pi v^-(x) dx, \end{aligned}$$

or

$$\begin{aligned} & \forall v \in \text{Ker}(-\Delta_p - \lambda) \setminus \{0\} : \\ & (p-1) \int_0^\pi h(x)v(x) dx > \overline{F(+\infty)} \int_0^\pi v^+(x) dx + \underline{F(-\infty)} \int_0^\pi v^-(x) dx, \end{aligned}$$

where

$$v^+ := \max\{0, v\}, \quad v^- := \min\{0, v\}.$$

Then the BVP (9) has at least one weak solution $u \in W_0^{1,p}(0, \pi)$.

³ All eigenvalues are described by the equalities

$$\lambda_k := \left(\frac{k\pi_p}{\pi} \right)^p = k^p \lambda_1, \quad k \in \mathbb{N},$$

where

$$\pi_p := 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{ds}{(1-s^p)^{\frac{1}{p}}}.$$

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