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$L^p$-THEORY OF THE NAVIER-STOKES FLOW IN THE EXTERIOR OF A MOVING OR ROTATING OBSTACLE

M. GEISSERT AND M. HIEBER

ABSTRACT. In this paper we describe two recent approaches for the $L^p$-theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle.

1. Introduction

Consider a compact set $O \subset \mathbb{R}^n$, the obstacle, with boundary $\Gamma := \partial O$ of class $C^{1,1}$. Set $\Omega := \mathbb{R}^n \setminus O$. For $t > 0$ and a real $n \times n$-matrix $M$ we set

$$\Omega(t) := \{y(t) = e^{tM}x, x \in \Omega\} \text{ and } \Gamma(t) := \{y(t) = e^{tM}x, x \in \Gamma\}.$$ 

Then the motion past the moving obstacle $O$ is governed by the equations of Navier-Stokes given by

$$\begin{align*}
\partial_t w - \Delta w + w \cdot \nabla w + \nabla q &= 0, & \text{in } \Omega(t) \times \mathbb{R}_+ , \\
\nabla \cdot w &= 0, & \text{in } \Omega(t) \times \mathbb{R}_+ , \\
w(y,0) &= w_0(y), & \text{in } \Omega .
\end{align*}$$

(1)

Here $w = w(y,t)$ and $q(y,t)$ denote the velocity and the pressure of the fluid, respectively. The boundary condition on $\Gamma(t)$ is the usual no-slip boundary condition. Quite a few articles recently dealt with the equation above, see [2], [3], [4], [5], [6], [8], [10], [11], [15], [16].

In this paper, we describe two approaches to the above equations for the $L^p$-setting where $1 < p < \infty$. The basic idea for both approaches is to transfer the problem given on a domain $\Omega(t)$ depending on $t$ to a fixed domain. The first transformation described in the following Section 2 yields additional terms in the equations which are of Ornstein-Uhlenbeck type. We shortly describe the techniques used in [15] and [12] in order to construct a local mild solution of (1).

In contrast to the first transformation, the second one, inspired by [17] and [6], allows to invoke maximal $L^p$-estimates for the classical Stokes operator in exterior domains and like this we obtain a unique strong solution to (1). This approach is described in section 3.

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2. Mild solutions

In this section we construct mild solutions to the Navier-Stokes problem (1). To do this we first transform the equations (1) to a fixed domain. Let \( \Omega, \Omega(t) \) and \( \Gamma(t) \) be as in the introduction and suppose that \( M \) is unitary. Then by the change of variables \( x = e^{-tM}y \) and by setting \( v(x, t) = e^{-tM}w(e^{tM}x, t) \) and \( p(x, t) = q(e^{tM}x, t) \) we obtain the following set of equations defined on the fixed domain \( \Omega \):

\[
\begin{aligned}
\partial_t v - \Delta v + v \cdot \nabla v - Mx \cdot \nabla v + Mv + \nabla p &= 0, & \text{in } \Omega \times \mathbb{R}_+,
\nabla \cdot v &= 0, & \text{in } \Omega \times \mathbb{R}_+,

v(x, t) &= Mx, & \text{on } \Gamma \times \mathbb{R}_+,

v(x, 0) &= w_0(x), & \text{in } \Omega.
\end{aligned}
\]

Note that the coefficient of the convection term \( Mx \cdot \nabla u \) is unbounded, which implies that this term cannot be treated as a perturbation of the Stokes operator.

This problem was first considered by Hishida in \( L^p(\Omega) \) for \( \Omega \subset \mathbb{R}^3 \) and \( Mx = \omega \times x \) with \( \omega = (0, 0, 1)^T \) in [15] and [16]. The \( L^p \)-theory was developed by Heck and the authors in [12] even for general \( M \).

We will construct mild solutions for \( w_0 \in L^p(\Omega) \), \( p \geq n \), to the problem (2) with Kato's iteration (see [18]).

The starting point is the linear problem

\[
\begin{aligned}
\partial_t u - \Delta u - Mx \cdot \nabla u + Mu + b \cdot \nabla u + u \cdot \nabla b + \nabla p &= 0, & \text{in } \Omega \times \mathbb{R}_+, \\
\nabla \cdot u &= 0, & \text{in } \Omega \times \mathbb{R}_+, \\
u(x, t) &= 0, & \text{on } \Gamma \times \mathbb{R}_+, \\
u(x, 0) &= w_0(x), & \text{in } \Omega,
\end{aligned}
\]

where \( b \in C_c^\infty(\overline{\Omega}) \). The additional term \( b \cdot \nabla u + u \cdot \nabla b \) simplifies the treatment of the Navier-Stokes problem (see (11) below). We will first show that the solution of (3) is governed by a \( C_0 \)-semigroup on \( L^p_\sigma(\Omega) \). More precisely, let \( L_{\Omega,b} \) be defined by

\[
\begin{aligned}
L_{\Omega,b} u &= P_\Omega L_b u \\
D(L_{\Omega,b}) &= \{ u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p(\Omega) : Mx \cdot \nabla u \in L^p(\Omega) \},
\end{aligned}
\]

where \( L_b u := \Delta u + Mx \cdot \nabla u - Mu + b \cdot \nabla u + u \cdot \nabla b \). Then the following theorem is proved in [12].

**Theorem 2.1.** Let \( 1 < p < \infty \) and let \( \Omega \subset \mathbb{R}^n \) be an exterior domain with \( C^{1,1} \)-boundary. Assume that \( \text{tr} M = 0 \) and \( b \in C_c^\infty(\overline{\Omega}) \). Then the operator \( L_{\Omega,b} \) generates a \( C_0 \)-semigroup \( T_{\Omega,b} \) on \( L^p_\sigma(\Omega) \).

**Sketch of the proof.** The proof is divided into several steps. First it is shown that \( L_{\Omega,b} \) is the generator of an \( C_0 \)-semigroup \( T_{\Omega,b} \) on \( L^p_\sigma(\Omega) \). Then a-priori \( L^p \)-estimates for \( T_{\Omega,b} \) are proved. Once we have shown this we can easily define a consistent family of semigroups \( T_{\Omega,b} \) on \( L^p_\sigma(\Omega) \) for \( 1 < p < \infty \). In the last step the generator of \( T_{\Omega,b} \) on \( L^p_\sigma(\Omega) \) is identified to be \( L_{\Omega,b} \).
We start by showing that $L_{Ω,b}$ is the generator of a $C_0$-semigroup on $L^2_b(Ω)$. Choose $R > 0$ such that $supp b ∪ Ω^c ⊂ B_R(0) = \{x ∈ ℜ^n : |x| < R\}$. We then set

$$D = Ω ∩ B_{R+5}(0),$$

$$K_1 = \{x ∈ Ω : R < |x| < R + 3\},$$

$$K_2 = \{x ∈ Ω : R + 2 < |x| < R + 5\}.$$

Denote by $B_i$ for $i \in \{1, 2\}$ Bogovskiǐ’s operator (see [1], [9, Chapter III.3], [13]) associated to the domain $K_i$ and choose cut-off functions $φ, η ∈ C^∞(ℜ^n)$ such that $0 ≤ φ, η ≤ 1$ and

$$φ(x) = \begin{cases} 0, & |x| ≤ R + 1, \\ 1, & |x| ≥ R + 2, \end{cases} \quad \text{and} \quad η(x) = \begin{cases} 1, & |x| ≤ R + 3, \\ 0, & |x| ≥ R + 4. \end{cases}$$

For $f ∈ L^p_b(Ω)$ we denote by $f^R$ the extension of $f$ by 0 to all of $ℜ^n$. Then, since $C^∞_{c,φ}(Ω)$ is dense in $L^p_b(Ω)$, $f^R ∈ L^p_b(ℜ^n)$. Furthermore, we set $f^D = ηf - B_2((∇η)f)$. Since $∫_{K_2}(∇η)f = 0$ it follows from [9, Chapter III.3] that $f^D ∈ L^p_b(D)$.

By the perturbation theorem for analytic semigroups there exists $ω_1 ≥ 0$ such that for $λ > ω_1$ there exist functions $u^D_λ$ and $p^D_λ$ satisfying the equations

$$(λ - L_b)u^D_λ + ∇p^D_λ = f^D, \quad \text{in } D × ℜ_+,$$

$$∇ · u^D_λ = 0, \quad \text{in } D × ℜ_+,$$

$$u^D_λ = 0, \quad \text{on } ∂D × ℜ_+.$$

(4)

Moreover, by [14, Lemma 3.3 and Prop. 3.4], there exists $ω_2 ≥ 0$ such that for $λ > ω_2$ there exists a function $u^R_λ$ satisfying

$$(λ - L_0)u^R_λ = f^R, \quad \text{in } ℜ^n × ℜ_+,$$

$$∇ · u^R_λ = 0, \quad \text{in } ℜ^n × ℜ_+.$$ 

(5)

For $λ > \max\{ω_1, ω_2\}$ we now define the operator $U_λ : L^p_b(Ω) → L^p_b(Ω)$ by

$$U_λf = φu^R_λ + (1 - φ)u^D_λ + B_1(∇φ(u^R_λ - u^D_λ)),$$

(6)

where $u^R_λ$ and $u^D_λ$ are the functions given above, depending of course on $f$. By definition, we have

$$U_λf ∈ \{v ∈ W^{2,p}(Ω) ∩ W^{1,p}_0(Ω) ∩ L^p_b(Ω) : Mx · ∇v ∈ L^p_b(Ω)\}.$$ 

(7)

Setting $P_λf = (1 - φ)p^D_λ$, we verify that $(U_λf, P_λf)$ satisfies

$$∀ x ∈ D \text{ such that } \{x\} ∈ \lambda K_{2i}.$$ 

(8)

where $T_λ$ is given by

$$T_λf = -2(∇φ)∇(u^R_λ - u^D_λ) - (∆φ + Mx · (∇φ))(u^R_λ - u^D_λ) + (∇φ)p^D_λ + \frac{1}{2}(λ - Δ - Mx · ∇ + M)B_1(∇φ(u^R_λ - u^D_λ)).$$ 

(9)
It follows from [12, Lemma 4.4] that for $\alpha \in (0, \frac{1}{2p'})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, there exists a strongly continuous function $H : (0, \infty) \to \mathcal{L}(L^p_\sigma(\Omega))$ satisfying
\[
\|H(t)\|_{\mathcal{L}(L^p_\sigma(\Omega))} \leq C t^{\alpha-1} e^{\omega t}, \quad t > 0
\]
for some $\omega \geq 0$ and $C > 0$ such that $\lambda \mapsto P_\lambda T_\lambda$ is the Laplace Transform of $H$.
We thus easily calculate
\[
\|P_\lambda T_\lambda\|_{\mathcal{L}(L^p_\sigma(\Omega))} \leq C\lambda^{-\alpha}, \quad \lambda > \omega.
\]
Therefore, $R_\lambda := U_\lambda \sum_{j=0}^{\infty} (P_\lambda T_\lambda)^j$ exists for $\lambda$ large enough and $(\lambda - L_0)R_\lambda f = f$ for $f \in L^2_\sigma(\Omega)$. Since $L_{\Omega,b}$ is dissipative in $L^2_\sigma(\Omega)$, $L_{\Omega,b}$ generates a $C_0$-semigroup $T_{\Omega,b}$ on $L^2_\sigma(\Omega)$. Moreover, we have the representation
\[
T_{\Omega,b}(t)f = \sum_{n=0}^{\infty} T_n(t)f, \quad f \in L^2_\sigma(\Omega),
\]
where $T_n(t) := \int_0^t T_{n-1}(t-s) H(s) \, ds$ for $n \in \mathbb{N}$ and
\[
T_0(t) = \varphi T_R(t) f^R + (1-\varphi) T_{D,b}(t) f^D + B_1((\nabla \varphi)(T_R(t)f^R - T_{D,b}(t)f^D)), \quad t \geq 0.
\]
Here $T_R$ denotes the semigroup on $L^p_\sigma(\mathbb{R}^n)$ generated by $L_{b} - 0$ and $T_{D,b}$ denotes the semigroup on $L^p_\sigma(D)$ generated by $L_{\Omega,b}$. Note that $\lambda \mapsto U_\lambda$ is the Laplace Transform of $T_\lambda$. Since the right hand side of the representation (9) is well defined and exponentially bounded in $L^p_\sigma(\Omega)$ by [12, Lemma 4.6], we can define a family of consistent semigroups $T_{\Omega,b}$ on $L^p(\Omega)$ for $1 < p < \infty$. Finally, the generator of $T_{\Omega,b}$ on $L^p(\Omega)$ is $L_{\Omega,b}$ which can be proved by using duality arguments (cf. [12, Theorem 4.1]).

Remark 2.2. (a) The semigroup $T_{\Omega,b}$ is not expected to be analytic since, by [16, Proposition 3.7], the semigroups $T_{\b|$ $\Omega|}$ in $\mathbb{R}^3$ is not analytic.
(b) As the cut-off function $\varphi$ is used for the localization argument similarly to [15] the purpose of $\eta$ is to ensure that $f_D \in L^p_\sigma(\Omega)$. This is essential to establish a decay property in $\lambda$ for the pressure $P_\lambda^D$ (cf. [12, Lemma 3.5]) and $T_\lambda$.
(c) The crucial point for a-priori $L^p$-estimates for $T_{\Omega,b}$ on $L^p_\sigma(\Omega)$ is the existence of $H$ satisfying (8).

Since $L^p-L^q$ smoothing estimates for $T_R$ and $T_{D,b}$ follow from [14, Lemma 3.3 and Prop. 3.4] and [12, Prop. 3.2], the representation of the semigroup $T_{\Omega,b}$ given by (9) and estimates for sums of convolutions of this type (cf. [12, Lemma 4.6]) yield the following proposition.

Proposition 2.3. Let $1 < p < q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$-boundary. Assume that $\text{tr} M = 0$ and $b \in C^\infty_c(\Omega)$. Then there exist constants $C > 0, \omega \geq 0$ such that for $f \in L^p_\sigma(\Omega)$
\begin{itemize}
  \item[(a)] $\|T_{\Omega,b}(t)f\|_{L^q_\sigma(\Omega)} \leq C t^{-\frac{q}{p}} \left(\frac{1}{p} - \frac{1}{q}\right) e^{\omega t} \|f\|_{L^p_\sigma(\Omega)}$, \quad $t > 0$,
  \item[(b)] $\|
abla T_{\Omega,b}(t)f\|_{L^p(\Omega)} \leq C t^{-\frac{q}{p}} e^{\omega t} \|f\|_{L^p_\sigma(\Omega)}$, \quad $t > 0$.
\end{itemize}
Moreover, for $f \in L^p_\sigma(\Omega)$
\begin{itemize}
  \item[(a)] $\|t^{-\frac{q}{p}} \left(\frac{1}{p} - \frac{1}{q}\right) T_{\Omega,b}(t)f\|_{L^p_\sigma(\Omega)} \to 0$, \quad \text{as} \quad t \to 0$,
In order to construct a mild solution to (2) choose \( \zeta \in C^\infty_c(\mathbb{R}^n) \) with \( 0 \leq \zeta \leq 1 \) and \( \zeta = 1 \) near \( \Gamma \). Further let \( K \subset \mathbb{R}^n \) be a domain such that \( \text{supp} \nabla \zeta \subset K \). We then define \( b : \mathbb{R}^n \to \mathbb{R}^n \) by
\[
(10) \quad b(x) := \zeta Mx - B_K((\nabla \zeta)Mx),
\]
where \( B_K \) is Bogovskiǐ's operator associated to the domain \( K \). Then \( \text{div} \ b = 0 \) and \( b(x) = Mx \) on \( \Gamma \). Setting \( u := v - b \), it follows that \( u \) satisfies
\[
\partial_t u - L \bxu + \nabla p = F, \quad u(0) = u_0 - b, \quad \text{on } \Omega \times (0, T),
\]
with \( \nabla \cdot (u_0 - b) = 0 \) in \( \Omega \) and \( F = -\Delta b - Mx \cdot \nabla b + Mb + b \cdot \nabla b \), provided \( u \) satisfies (2). Applying the Helmholtz projection \( P_\Omega \) to (11), we may rewrite (11) as an evolution equation in \( L^p_\sigma(\Omega) \):
\[
(12) \quad u' - L_{\Omega,b}u + P_\Omega(u \cdot \nabla u) = P_\Omega F, \quad u(0) = u_0 - b.
\]

Then the main result of [12] is the following theorem.

**Theorem 2.4.** Let \( n \geq 2, n \leq p \leq q < \infty \) and let \( \Omega \subset \mathbb{R}^n \) be an exterior domain with \( C^{1,1} \)-boundary. Assume that \( \text{tr} \ M = 0 \) and \( b \in C^\infty_c(\bar{\Omega}) \) and \( u_0 - b \in L^p_\sigma(\Omega) \). Then there exist \( T_0 > 0 \) and a unique mild solution \( u \) of (12) such that
\[
t \mapsto t^{\frac{n}{p} \left( \frac{1}{p} - \frac{1}{q} \right)} u(t) \in C \left( [0, T_0]; L^p_\sigma(\Omega) \right),
\]
and
\[
t \mapsto t^{\frac{n}{p} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2}} \nabla u(t) \in C \left( [0, T_0]; L^q(\Omega) \right).
\]

### 3. Strong solutions

In this section we construct strong solutions to problem (1) for \( \Omega \subset \mathbb{R}^n, n \geq 2 \) and \( \text{tr} \ M = 0 \). The main difference to the method presented in the previous section is another change of variables. Indeed, we construct a change of variables which coincides with a simple rotation in a neighborhood of the rotating body but it equals to the identity operator far away from the rotating body. More precisely,
let $X(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ denote the time dependent vector field satisfying
\[
\frac{\partial X}{\partial t}(y, t) = -b(X(y, t)), \quad y \in \mathbb{R}^n, \quad t > 0,
\]
\[
X(y, 0) = y, \quad y \in \mathbb{R}^n,
\]
where $b$ is as in (10). Similarly to [6, Lemma 3.2], the vector field $X(\cdot, t)$ is a $C^\infty$-diffeomorphism form $\Omega$ onto $\Omega(t)$ and $X \in C^\infty([0, \infty) \times \mathbb{R}^n)$. Let us denote the inverse of $X(\cdot, t)$ by $Y(\cdot, t)$. Then, $Y \in C^\infty([0, \infty) \times \mathbb{R}^n)$. Moreover, it can be shown that for any $T > 0$ and $|a| + k > 0$ there exists $C_{k,a,T} > 0$ such that
\[
(13) \quad \sup_{y \in \mathbb{R}^n, 0 \leq t \leq T} \left| \frac{\partial^k \partial^\alpha}{\partial t^k \partial y^\alpha} X(y, t) \right| + \sup_{x \in \mathbb{R}^n, 0 \leq t \leq T} \left| \frac{\partial^k \partial^\alpha}{\partial t^k \partial x^\alpha} Y(x, t) \right| \leq C_{k,a,T}.
\]
Setting
\[
v(x, t) = J_X(Y(x, t), t)w(Y(x, t), t), \quad x \in \Omega, \quad t \geq 0,
\]
where $J_X$ denotes the Jacobian of $X(\cdot, t)$ and
\[
p(x, t) = q(Y(x, t), t), \quad x \in \Omega, \quad t \geq 0,
\]
similarly to [6, Prop. 3.5] and [17], we obtain the following set of equations which are equivalent to (1).
\[
\partial_t v - \mathcal{L}v + \mathcal{M}v + \mathcal{N}v + \mathcal{G}p = 0, \quad \text{in } \Omega \times \mathbb{R}_+,
\]
\[
v(x, t) = Mx, \quad \text{on } \Gamma \times \mathbb{R}_+, \\
v(x, 0) = w_0(x), \quad \text{in } \Omega.
\]
Here
\[
(\mathcal{L}v)_i = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left( g^{jk} \frac{\partial v_i}{\partial x_k} \right) + 2 \sum_{j,k,l=1}^{n} g^{kl} \Gamma_{jk}^{l} \frac{\partial v_i}{\partial x_l},
\]
\[
\quad + \sum_{j,k,l=1}^{n} \left( \frac{\partial}{\partial x_k} (g^{kl} \Gamma_{ji}^{k}) + \sum_{m=1}^{n} g^{kl} \Gamma_{jm}^{k} \Gamma_{km}^{i} \right) v_j,
\]
\[
(\mathcal{N}v)_i = \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} + \sum_{j,k=1}^{n} \Gamma_{jk}^{i} v_j v_k,
\]
\[
(\mathcal{M}v)_i = \sum_{j=1}^{n} \frac{\partial X_j}{\partial t} \frac{\partial v_i}{\partial x_j} + \sum_{j,k=1}^{n} \left( \Gamma_{jk}^{i} \frac{\partial X_k}{\partial t} + \frac{\partial X_i}{\partial x_k} \frac{\partial^2 Y_k}{\partial x_j \partial t} \right) v_j,
\]
\[
(\mathcal{G}p)_i = \sum_{j=1}^{n} g^{ij} \frac{\partial p}{\partial x_j}
\]
with
\[
g^{ij} = \sum_{k=1}^{n} \frac{\partial X_i}{\partial y_k} \frac{\partial X_j}{\partial y_k}, \quad g_{ij} = \sum_{k=1}^{n} \frac{\partial Y_k}{\partial x_i} \frac{\partial Y_j}{\partial x_k}
\]
and
\[
\Gamma_{ij}^{k} = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left( \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{ij}}{\partial x_l} \right).
\]
The obvious advantage of this approach is that we do not have to deal with an unbounded drift term since all coefficients appearing in $\mathcal{L}, \mathcal{N}, \mathcal{M}$ and $\mathcal{G}$ are smooth and bounded on finite time intervals by (13). However, we have to consider a non-autonomous problem. Setting $u = v - b$, we obtain the following problem with homogeneous boundary conditions which is equivalent to (14).

$$\begin{align*}
\partial_t u - \mathcal{L}u + \mathcal{M}u + \mathcal{N}u + \mathcal{B}u + \mathcal{G}p &= F_b, & \text{in } \Omega \times \mathbb{R}_+, \\
\nabla \cdot u &= 0, & \text{in } \Omega \times \mathbb{R}_+, \\
\nabla u &= 0, & \text{on } \Gamma \times \mathbb{R}_+, \\
u(x,0) &= w_0(x) - b(x), & \text{in } \Omega.
\end{align*}$$

(15)

Here,

$$\begin{align*}
(Bu)_i &= \sum_{j=1}^n \left( u_j \frac{\partial b_j}{\partial x_j} + b_j \frac{\partial u_i}{\partial x_j} \right) + 2 \sum_{j,k=1}^n \Gamma_{j,k} u_j b_k, & F_b = \mathcal{L}b - \mathcal{M}b - \mathcal{N}b.
\end{align*}$$

Since $g^{ij}$ is smooth and $g^{ij}(\cdot, 0) = \delta_{ij}$ by definition, it follows from (13) that

$$\|g^{ij}(\cdot, t) - \delta_{ij}\|_{L^\infty(\Omega)} \to 0, \quad t \to 0.
$$

(16)

In other words, $\mathcal{L}$ is a small perturbation of $\Delta$ and $\mathcal{G}$ is a small perturbation of $\nabla$ for small times $t$. This motivates to write (15) in the following form.

$$\begin{align*}
\partial_t u - \Delta u + \nabla p &= F(u, p), & \text{in } \Omega \times \mathbb{R}_+, \\
\nabla \cdot u &= 0, & \text{in } \Omega \times \mathbb{R}_+, \\
u &= 0, & \text{on } \Gamma \times \mathbb{R}_+, \\
u(x,0) &= w_0(x) - b(x), & \text{in } \Omega.
\end{align*}$$

(17)

where $F(u, p) := (\mathcal{L} - \Delta)u - \mathcal{M}u - \mathcal{N}u + (\nabla - \mathcal{G})p - \mathcal{B}u + \mathcal{G}b$. We will use maximal $L^p$-regularity of the Stokes operator and a fixed point theorem to show the existence of a unique strong solution $(u, p)$ of (15). More precisely, let

$$X_T^{p,q} := W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; D(A_q)) \times L^p(0, T; \tilde{W}^{1,p}(\Omega)),$$

where $D(A_q) := W^{2,q}(\Omega) \cap W^{1,q}_d(\Omega) \cap L^q(\Omega)$ is the domain of the Stokes operator. Then, by maximal $L^p$-regularity of the Stokes operator, Hölder’s inequality and Sobolev’s embedding theorems $\Phi : X_T^{p,q} \to X_T^{p,q}, \Phi((\hat{u}, \hat{p})) := (u, p)$ where $(u, p)$ is the unique solution of

$$\begin{align*}
\partial_t u - \Delta u + \nabla p &= F(\hat{u}, \hat{p}), & \text{in } \Omega \times (0, T) \\
\nabla \cdot u &= 0, & \text{in } \Omega \times (0, T) \\
u &= 0, & \text{on } \Gamma \times (0, T), \\
u(x,0) &= w_0(x) - b(x), & \text{in } \Omega,
\end{align*}$$

is well-defined for $1 < p, q < \infty$ with $\frac{2}{q} + \frac{1}{p} < \frac{3}{2}$ and $T > 0$. Here, the restriction on $p$ and $q$ comes from the nonlinear term $\mathcal{N}$.

Finally, let $X_T^{p,q}_\delta := \{(u, p) \in X_T^{p,q} : \|\|(u, p) - (\hat{u}, \hat{p})\|_{X_T^{p,q}} \leq \delta, u(0) = w_0 - b\}$ with $(\hat{u}, \hat{p}) = \Phi(\Phi(0, 0))$. Then by (16), Hölder’s inequality and Sobolev’s embedding theorems, it can be shown that for small enough $\delta > 0$ and $T > 0$, $\Psi|_{X_T^{p,q}_\delta}$ is a contraction.

We summarize our considerations in the next theorem which is proved in [7]. Note that the cases $n = 2, 3$ and $p = q = 2$ were already proved in [6].
Theorem 3.1. Let $1 < p, q < \infty$ such that $\frac{n}{2q} + \frac{1}{p} < \frac{3}{2}$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$-boundary. Assume that $\text{tr} M = 0$ and that $w_0 - b \in (L^q_2(\Omega), D(A_q))_{\frac{1}{1-p}, p}$. Then there exist $T > 0$ and a unique solution $(u, p) \in X_{p,q}^T$ of problem (15).

References