José M. Arrieta; Aníbal Rodríguez-Bernal
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BLOW UP VERSUS GLOBAL BOUNDEDNESS OF SOLUTIONS OF REACTION DIFFUSION EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS∗
JOSE M. ARRIETA† AND ANIBAL RODRIGUEZ-BERNAL‡

Abstract. In this paper we analyze the behavior of solutions of reaction-diffusion equations with nonlinear boundary conditions of the type (1.1). We show that if \( f(x,u) = -\beta_0 u^p \) and \( g(x,u) = u^q \) in a neighborhood of a point \( x_0 \in \Gamma_N \), then

i) for the case \( q > 1 \), if \( p + 1 < 2q \) or if \( p + 1 = 2q \) and \( \beta_0 < q \), then blow up in finite time at \( x_0 \) occurs.

ii) for the case \( p > 1 \) if \( p + 1 > 2q \) or if \( p + 1 = 2q \) and \( \beta_0 > q \) then any solution is globally bounded around the point \( x_0 \).

Key words. reaction-diffusion, nonlinear boundary conditions, blow-up, boundedness

1. Introduction. We consider the following reaction diffusion equation with nonlinear boundary conditions in a smooth \( C^2 \) domain \( \Omega \subset \mathbb{R}^N \),

\[
\begin{aligned}
    u_t - \Delta u &= f(x,u) \quad \text{in } \Omega \\
    u &= 0 \quad \text{on } \Gamma_D \\
    \frac{\partial u}{\partial \vec{n}} &= g(x,u) \quad \text{on } \Gamma_N \\
    u(0,x) &= u_0(x) \geq 0 \quad \text{in } \Omega 
\end{aligned}
\]

(1.1)

where \( \Gamma = \partial \Omega = \Gamma_D \cup \Gamma_N \) is a regular disjoint partition of the boundary of \( \Omega \) and \( f \) and \( g \) are suitably smooth functions of \( (x,u) \). The subindices \( D \) and \( N \) on \( \Gamma \) indicate the part of the boundary with Dirichlet and Neumann type condition, respectively. We are interested in nonnegative solutions of (1.1) so we will assume

\[ f(x,0) \geq 0, \quad \text{for all } x \in \Omega, \quad g(x,0) \geq 0 \quad \text{for all } x \in \Gamma_N \]

We want to obtain local conditions on the nonlinearities \( f \) and \( g \), which will be imposed in a neighborhood of a point \( x_0 \in \Gamma_N \), that guarantee that either i) there exists initial conditions with support in a neighborhood of \( x_0 \) such that the “proper solution” starting at this initial condition blows up at \( x_0 \) or that ii) for all initial data \( u_0 \in L^\infty(\Omega) \) the “proper solution” starting at \( u_0 \) is bounded in a neighborhood of \( x_0 \) for all times \( t \geq 0 \). We refer to [4, 8, 9] for the concept of proper solution.

Notice that if \( f(x,u) \) behaves like \( u^p \) locally around certain point \( z \in \Omega \) and \( p > 1 \), then, by comparison with the Dirichlet problem in a neighborhood of \( z \) and using that the superlinear nonlinearity \( u^p \) is explosive we get that, regardless of the behavior of \( g \), we have initial conditions that blow-up in finite time. On the other hand, if \( f(x,u) \) behaves like \(-u^p \) and \( g(x,u) \) behaves like \(-u^q \) throughout the whole domain, then both

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†Depto. de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain (arrieta@mat.ucm.es)
‡Depto. de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain (arober@mat.ucm.es)
nonlinearities are dissipative and we have global existence and boundedness of solutions. The most interesting case is when \( f(x, u) \) is a dissipative nonlinearity of the form \( -\beta_0 u^p \) and \( g(x, u) \) is an explosive nonlinearity of the form \( u^q \). This two mechanisms are in competition and it seems clear that the relative size of \( p \), \( q \) and \( \beta_0 \) will determine the relative strength of both mechanisms.

Actually, in the pioneer work of [6] they treated the one dimensional case, say \( \Omega = (0, 1) \), with \( f(x, u) = -\beta_0 u^p \), \( g(x, u) = u^q \) and \( \Gamma_D = \emptyset \) and they already obtained that the critical relations are \( p + 1 \) vs. \( 2q \) and if \( p + 1 = 2q \) then \( \beta_0 \) vs. \( q \), in the sense that if \( p + 1 < 2q \) or \( p + 1 = 2q \) and \( \beta_0 < q \) then blow-up is produced and if \( p + 1 > 2q \) or \( p + 1 = 2q \) and \( \beta_0 > q \) then the solutions are globally bounded. They also treated the very delicate case where \( p + 1 = 2q \) and \( \beta_0 = q \). They actually showed that the solutions were defined for all time \( t > 0 \) but the phenomenon of infinite time blow-up was present.

Later on, in [13, 14], they treated the case of arbitrary dimension and obtained that if \( \Gamma_D = \emptyset \) and the nonlinearities \( f \) and \( g \) that behave for \( u \) large as \( f \sim -\beta_0 u^p \) and \( g \sim u^q \), then blow-up is produced if \( p + 1 < 2q \) or if \( p + 1 = 2q \) and \( \beta_0 < q \). Also, they showed that if \( p + 1 > 2q \) or if \( p + 1 = 2q \) and \( \beta_0 \) is large enough, then the solutions are globally bounded. Also, in [1] they studied the porous medium equation in any dimension and as a particular case they considered the equation (1.1) with \( \Gamma_D = \emptyset \), \( f(x, u) = -\beta_0 u^p \) and \( g(x, u) = u^q \). They showed that if \( p + 1 < 2q \) or \( p + 1 = 2q \) and \( \beta_0 < q \) then blow-up is produced and if \( p + 1 > 2q \) of \( p + 1 = 2q \) and \( \beta_0 > q \) then the solutions are globally bounded.

With all these works it is clear that the critical relations that mark the line between blow-up and boundedness are given by \( p + 1 \) vs. \( 2q \) and in case \( p + 1 = 2q \), \( \beta_0 \) vs. \( q \). These works have a common characteristic and it is that the nonlinear boundary condition is imposed in the whole domain, \( \Gamma_D = \emptyset \) and the construction of sub or super solutions is done for the whole domain. Hence, the balances between \( f \) and \( g \) need to hold throughout the domain to obtain the result and both, the blow-up and the boundedness result are global in space. In particular, none of them can treat the case as in the equation (4.1) where \( p + 1 = 2q \) but in some part of the boundary the relation is \( \beta_0 > q \) and in other part the relation is \( \beta_0 < q \) or even when \( \Gamma_D \neq \emptyset \).

In this paper we will prove that both mechanisms (dissipativeness vs. blow-up) compete at a local level. Actually, we will show that if \( f(x, u) = -\beta_0 u^p \) and \( g(x, u) = u^q \) in a neighborhood of a point \( x_0 \in \Gamma_N \), then

i) for the case \( q > 1 \), if \( p + 1 < 2q \) or if \( p + 1 = 2q \) and \( \beta_0 < q \), then blow up in finite time at \( x_0 \) occurs, see Section 2.

ii) for the case \( p > 1 \) if \( p + 1 > 2q \) or if \( p + 1 = 2q \) and \( \beta_0 > q \) then any solution is globally bounded around the point \( x_0 \), see Section 3.

In Section 2 we analyze the first case and we refer to [3] for details. In Section 3 we consider the case ii) and we announce the results of [2]. In Section 4 we consider several important remarks and comments.

2. Localization of blow-up. In terms of characterizing the sizes of \( p \), \( q \) and \( \beta_0 \) that will produce blow-up we have:

**Proposition 2.1.** Let \( x_0 \in \Gamma_N \), \( p \geq 1 \), \( q > 1 \) and let \( R_0 > 0 \), \( M_0 > 0 \) such that

\[
\begin{align*}
 f(x, u) &\geq -\beta_0 u^p, \quad x \in B(x_0, R_0) \cap \Omega, \quad u \geq M_0, \\
 g(x, u) &\geq u^q, \quad x \in B(x_0, R_0) \cap \partial\Omega, \quad u \geq M_0.
\end{align*}
\]

(2.1)

If one of the two following conditions holds

i) \( p + 1 < 2q \) or
ii) $p + 1 = 2q$ and $\beta_0 < q$, then, there exists an initial condition $0 \leq u_0 \in L^\infty(\Omega)$ with support in a neighborhood of $x_0$ such that the proper minimal solution of (1.1) starting at $u_0$ blows up in finite time at the point $x_0$.

Proof. Let us provide a proof of ii). Actually this case is more critical than i).

In order to simplify, consider that $x_0 = 0 \in \Gamma_N$ and that the outward normal vector at $x_0 = 0$ is given by $\vec{n}(0) = (0, \ldots, 0, -1)$. Let $R, \delta > 0$ be small numbers and $y_R = x_0 + R\vec{n}(x_0) = (0, \ldots, 0, -R)$ with the property that $B(y_R, R) \cap \Omega = \emptyset$ and that $B(y_R, R + \delta) \subset B(0, R_0/2)$. The fact that the domain has a $C^2$ boundary, guarantees that this construction can be done. See Fig. 2.1.

![Fig. 2.1. The domain $\Omega$ near $x_0$.](image)

We will construct a function $z(t, x)$ which will be radially symmetric around $y_R$, increasing in time and that it will be a subsolution of (1.1) locally around the point $x_0$. For this, define for $a \geq 1$, the function $\psi_0(t)$ as the solution of the problem

$$\begin{cases}
    \psi' = \psi^q, \\
    \psi(0) = a.
\end{cases}$$

(2.2)

![Fig. 2.2. The solution of Equation (2.2).](image)
Solving this equation, we get that $\psi_\alpha(t) = \frac{E}{(T_a - t)^{\frac{1}{q-1}}}$ for $-\infty < t < T_a$ with $E = \frac{1}{(q - 1)\alpha^{q-1}}$ and $T_a = \frac{1}{(q - 1)\alpha^{q-1}}$. Observe that, since $a \geq 1$ and $q > 1$, $T_a \leq 1/(q - 1)$ and that $T_a \to 0$ as $a \to +\infty$. Notice also that $\psi_\alpha(t) \leq E/(-t)^{1/(q-1)}$ for any $t < 0$ and any $a \geq 1$.

We define $z_\alpha(t, x) = \psi_\alpha(t + R - |x - y_R|)$ for $x \in \mathbb{R}^N \setminus B(y_R, R)$, $0 \leq t < T_a$, see Fig. 2.3.

Direct computations show that $\frac{\partial z_\alpha}{\partial n} \leq \frac{E}{2\delta a}$ for $x \in \Gamma_1$ and $0 < t < T_a$ and $\frac{\partial z_\alpha}{\partial t} - \Delta z_\alpha \leq (1 + \frac{N-1}{R} - qz_\alpha^{q-1})z_\alpha^q$ for $x \in \Omega \cap B(y_R, R + \delta)$ and $t \in (0, T_a)$. Notice that $z_\alpha$ is increasing in time and that $z_\alpha(t, x) \geq z_\alpha(0, x) = \psi_\alpha(R - |x - y_R|) = \psi_\alpha(-\delta) = \frac{E}{(T_a + \delta)^{\frac{1}{q-1}}} \to +\infty$ as $a \to +\infty$ and $\delta \to 0$, for $x \in \Omega \cap B(y_R, R + \delta)$. Hence, choosing $a_0$ large enough and $\delta_0$ small enough, we can guarantee, since $\beta_0 < q$, that for $a \geq a_0$ and $0 < \delta < \delta_0$, that $1 + \frac{N-1}{R} - qz_\alpha^{q-1} \leq -\beta_0z_\alpha^{q-2} = -\beta_0z_\alpha^q$ as long as $x \in \Omega \cap B(y_R, R + \delta)$ and $0 \leq t < T_a$.

In particular, we get
\begin{align} 
\begin{cases} 
\frac{\partial z_\alpha}{\partial t} - \Delta z_\alpha \leq -\beta_0z_\alpha^q, & x \in \Omega \cap B(y_R, R + \delta), \ t \in (0, T_a), \\
\frac{\partial z_\alpha}{\partial n} \leq z_\alpha^q, & x \in \Gamma_1 = \partial \Omega \cap B(y_R, R + \delta), \ t \in (0, T_a). 
\end{cases} \tag{2.3} 
\end{align}

Consider now a smooth initial condition $v_0 \in C^\infty(\Omega)$ such that $v_0 \equiv 0$ in $\Omega \setminus B(0, R_0)$ and $u_0 \geq \frac{2E}{\delta}$ in $\Omega \cap B(y_R, R + \delta)$. The solution of (1.1) starting at $u_0$ will satisfy that for a small time $T$ we will have that $u(x, t, v_0) \geq \frac{2E}{\delta}$ for $x \in \Gamma_0 \equiv \Omega \cap \partial B(y_R, R + \delta)$, $0 \leq t < T$. By monotonicity, for any $u_0 \geq v_0$ in $\Omega$, we will also have that the proper solution starting at $u_0$ will satisfy, $u(x, t, u_0) \geq \frac{E}{\delta}$ for $x \in \Gamma_0 \equiv \Omega \cap \partial B(y_R, R + \delta)$, $0 \leq t < T$.

In particular, let us choose $a > a_0$ with the property that $0 < T_a < T$ and let us choose $u_0$ such that $u_0(x) \geq v_0(x)$ and $u_0(x) \geq \psi_\alpha(-R) \geq z_\alpha(0, x)$ for $x \in \Omega \cap B(y_R, R + \delta)$. Hence, for $0 \leq t < T_a$ we have $z_\alpha(t, x) \leq \frac{E}{\delta} \leq u(x, t, u_0)$ for $x \in \Gamma_0$ and $z_\alpha(0, x) \leq u_0(x)$.
for \( x \in \Omega \cap B(y_R, R + \delta) \). That is, \( z_a \) satisfies,

\[
\begin{aligned}
\frac{\partial z_a}{\partial t} - \Delta z_a &\leq -\beta_0 z_a^p, \quad x \in \Omega \cap B(y_R, R + \delta), \ t \in (0, T_a), \\
\frac{\partial z_a}{\partial n} &\leq z_a^q, \quad x \in \Gamma_1 = \partial \Omega \cap B(y_R, R + \delta), \ t \in (0, T_a), \\
z_a(t, x) &\leq u(x, t, u_0), \quad x \in \Gamma_0, \ t \in (0, T_a), \\
z_a(0, x) &\leq u_0, \quad x \in \Omega \cap B(y_R, R + \delta),
\end{aligned}
\]  

(2.4)

which implies that \( z_a(t, x) \leq u(t, x, u_0) \) for all \( x \in \Omega \cap B(y_R, R + \delta) \) and \( t \in (0, T_a) \). The fact that \( z_a(T_a, x) \) blows up at \( x = 0 \) proves the result. \( \square \)

**Remarks.**

i) The time \( T_a \) does not need to be the classical blow-up time, that is, the time \( T_\infty \) for which the solution is classical for \( 0 < t < T_\infty \) and such that \( \|u(t, \cdot, u_0)\|_{L^\infty(\Omega)} \to +\infty \) as \( t \nearrow T_\infty \). We just can assure that \( T_\infty \leq T_a \).

ii) Observe that if for \( \alpha \in (0, T - T_a) \) we define the function \( w_\alpha(t, x) = z_a(t - \alpha, x) \)

defined for \( x \in \Omega \cap B(y_R, R + \delta) \) and \( t \in (\alpha, T_a + \alpha) \), then, we easily obtain that \( w_\alpha \) satisfies

\[
\begin{aligned}
\frac{\partial w_\alpha}{\partial t} - \Delta w_\alpha &\leq -\beta_0 w_\alpha^p, \quad x \in \Omega \cap B(y_R, R + \delta), \ t \in (\alpha, T_a + \alpha), \\
\frac{\partial w_\alpha}{\partial n} &\leq w_\alpha^q, \quad x \in \Gamma_1 = \partial \Omega \cap B(y_R, R + \delta), \ t \in (\alpha, \alpha + T_a), \\
w_\alpha(t, x) &\leq u(x, t, u_0), \quad x \in \Gamma_0, \ t \in (\alpha, \alpha + T_a), \\
w_\alpha(\alpha, x) &\leq u_0, \quad x \in \Omega \cap B(y_R, R + \delta).
\end{aligned}
\]  

(2.5)

The third inequality is obtained since for \( x \in \Gamma_0 \) we have \( w_\alpha(t, x) \leq \frac{E}{\delta + x} \leq u(x, t, u_0) \).

From (2.5) we obtain that \( w_\alpha(t, x) = z_a(t - \alpha, x) \leq u(t, x, u_0) \) for all \( \alpha \in (0, T - T_a) \).

This implies that for \( t \in (T_a, T) \) we have \( z_a(T_a, x) \leq u(t, x, u_0) \) which means that the solution \( u \) is “pinned” to the value \( \infty \) during the time \( T_a \leq t \leq T \).

iii) With some extra effort, see [3] for details, it is possible to show that the construction of PROPOSITION 2.1 can be performed in a neighborhood of \( x_0 \in \partial \Omega \). As a matter of fact the parameters, \( R, \delta, u_0, \delta_0 \), and the initial condition \( u_0 \) can be chosen the same for a small neighborhood \( \partial \Omega \cap B(x_0, \eta) \) for \( \eta > 0 \) small. This means that the proper solution \( u(t, x, u_0) \) will blow up, not only at \( x_0 \) but at \( B(x_0, \eta') \cap \partial \Omega \) for some small \( \eta' > 0 \), and it will remain “pinned” to the value \( \infty \) for a period of time \( T_a \leq t \leq T \).

**3. Localization of global boundedness.** In this section we present the results of [2] that, roughly speaking, say that if the complementary conditions of PROPOSITION 2.1 hold, also near a point \( x_0 \in \partial \Omega \), then the proper solution is bounded globally in time around this point \( x_0 \). As a matter of fact, we have

**PROPOSITION 3.1.** Let \( x_0 \in \Gamma_N \), \( p > 1 \), \( q \geq 1 \) and let \( R_0 > 0 \) and \( M_0 > 0 \) such that

\[
\begin{aligned}
f(x, u) &\leq -\beta_0 u^p, \quad x \in B(x_0, R_0) \cap \Omega, \quad u \geq M_0, \\
g(x, u) &\leq u^q, \quad x \in B(x_0, R_0) \cap \partial \Omega, \quad u \geq M_0.
\end{aligned}
\]  

(3.1)

If one of the two following conditions holds

i) \( p + 1 > 2q \) and \( \beta_0 > 0 \) or

ii) \( p + 1 = 2q \) and \( \beta_0 > q \).
then, for any initial condition \(0 \leq u_0 \in L^\infty(\Omega)\) the proper solution of (1.1) starting at \(u_0\) is bounded in a neighborhood of \(x_0\) in \(\bar{\Omega}\), for all \(t > 0\). That is, there exist \(\delta, M > 0\) such that

\[
\sup_{0 \leq t < \infty, x \in B(x_0, \delta) \cap \bar{\Omega}} u(t, x, u_0) \leq M. \tag{3.2}
\]

To prove the result, we construct appropriate super solutions locally around the point \(x_0 \in \Gamma_N\). As a matter of fact we extensively use the singular solutions of the following elliptic problem

\[
\begin{cases}
-\Delta z + \beta z^p = 0 & \text{in } B(0, R), \\
z(R) = +\infty,
\end{cases}
\]

and the fact that the asymptotics of this radial solution as \(r \to R\) is well understood, see [5, 12]. We refer to [2] for details on the proof of this result.

4. Concluding Remarks. We present in this section several important comments and remarks.

i) Both results are local in nature: if the conditions of Proposition 2.1 (resp. Proposition 3.1) hold in a neighborhood of certain point \(x_0 \in \partial \Omega\), then, independently of the behavior of the nonlinearities outside this neighborhood, we will have that blow-up (resp. global boundedness of solutions) occurs in the neighborhood of \(x_0\). In particular, from the control theory point of view it turns out that it is impossible to prevent blow-up (resp. to produce blow-up) in a neighborhood of a point of the boundary of the domain by modifying the equation somehow away from this point.

![Fig. 4.1. The domain of the example.](image)

ii) With an appropriate rescaling it is not difficult to see that if the local conditions of the nonlinearities \(f\) and \(g\) in Proposition 2.1 and Proposition 3.1 are of the type \(f(x, u) \sim -\beta_0 u^p\), \(g(x, u) \sim \alpha_0 u^q\), for \(x \in B(x_0, R_0) \cap \partial \Omega, u \geq M_0\), then, the condition \(\beta_0 < q\) (resp. \(\beta_0 > q\)) should be changed to \(\beta_0 > q\alpha_0^2\), (resp. \(\beta_0 < q\alpha_0^2\)).

iii) It is important to mention that the balances obtained for \(p, q\) and \(\beta_0\) are independent of the dimension of the space and even of the geometry of the domain.
iv) As an example, consider for instance the problem

\[ u_t - \nabla u = -\beta(x)u^p \quad \text{in } \Omega, \]

\[ \frac{\partial u}{\partial \vec{n}} = \alpha(x)u^q \quad \text{on } \partial\Omega, \]  

\[ u(0, x) = u_0(x) \geq 0 \quad \text{in } \Omega, \]  

(4.1)

with \( \beta \) and \( \alpha \) continuous functions, \( \beta(x) > 0 \) in \( \bar{\Omega} \) and \( \alpha(x) > 0 \) in \( \partial\Omega \), see Fig. 4.1.

Then if \( p + 1 = 2q > 2 \) and \( x_0 \in \partial\Omega \) with \( \frac{\beta(x_0)}{\alpha(x_0)^{\frac{1}{2}}} < q \) then from [3], there are initial conditions where blow up is produced near \( x_0 \), while if \( \frac{\beta(x_0)}{\alpha(x_0)^{\frac{1}{2}}} > q \), then from Theorem 2.1 above, for any initial condition \( u_0 \in L^\infty(\Omega) \) the proper minimal solution is bounded near \( x_0 \). Hence, we have the situation as in Fig. 4.1.

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