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CONVERGENCE IN EVOLUTIONARY VARIATIONAL INEQUALITIES WITH HYSTERESIS NONLINEARITIES*

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Abstract. Sufficient frequency-domain conditions for the convergence of solutions of evolutionary variational inequalities with hysteresis nonlinearities to stationary solutions are derived. The convergence is considered in non-standard chains of rigged Hilbert spaces. Monotonicity properties of operators are introduced with respect to the pairing between different spaces of such chains.

Key words. Variational inequality; hysteresis; convergence; frequency-domain conditions

AMS subject classifications. Primary 34G20, 44A05 Secondary 47D06

1. Introduction. We investigate the convergence of solutions of a class of evolutionary variational inequalities with hysteresis nonlinearities using absolute stability methods. For the ODE-case this was firstly done by V. A. Yakubovich in [14]. We generalize one of his results to variational inequalities. For the case of almost-periodic solutions this was done in [9]. In the absolute stability approach some properties of the hysteresis operator are characterized by quadratic constraints. Then the solvability of special operator Lur'e inequalities ([6]), introduced with respect to the "linear part" of the variational inequality and the quadratic constraints, is used to construct non-standard Gelfand riggings of a given Hilbert space. This allows us to describe monotonicity and regularity properties of operators with respect to the constructed chain and to show the convergence of solutions to equilibria in some energy-like spaces. As a consequence, our sufficient condition for convergence is satisfied for some variational inequalities (or variational equations) with operators which are not monotone or maximal monotone with respect to standard scalar products ([3, 4, 11]).

2. Non-standard Gelfand riggings generated by Lyapunov operators. Consider the Gelfand rigging of a real Hilbert space Y_0, i.e. a chain

$$Y_1 \subset Y_0 \subset Y_{-1} \tag{2.1}$$

in which Y_1 ("positive" space) and Y_{-1} ("negative" space) are further real Hilbert spaces and the inclusions are dense and continuous. Let $(\cdot,\cdot)_i$ and $\|\cdot\|_i, i = 1, 0, -1$, denote the scalar product and the norm in Y_i, respectively. Continuity of the inclusions means that there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$\|y\|_0 \le c_1 \|y\|_1 \ , \qquad \forall\, y \in Y_1 \tag{2.2}$$

and

$$c_2 \|u\|_{-1} \le \|u\|_0 \ , \qquad \forall\, u \in Y_0 \ . \tag{2.3}$$

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Suppose that the rigging (2.1)–(2.3) is realized in the following sense ([2, 13]). Assume that from the inclusion chain (2.1) only $Y_1 \subset Y_0$ is given and (2.2) is satisfied, for simplicity, with $c_1 = 1$. We introduce on Y_0 a second norm by

$$\|y\|_{-1} := \sup_{0 \neq \eta \in Y_1} \frac{|(y,\eta)_0|}{\|\eta\|_1} \tag{2.4}$$

and denote by Y_{-1} the completion of Y_0 with respect to this norm. Then Y_{-1} can be taken as third space in the Gelfand rigging (2.1) (see [2, 13]). This space can be considered as dual to Y_1 with respect to Y_0, i.e. when the duality of Y_1 and Y_{-1} is written in terms of Y_0. Extending by continuity the function $(u,v)_0$ onto $Y_{-1} \times Y_1$, we get the pairing between Y_{-1} and Y_1, i.e. the bilinear form $(\cdot,\cdot)_{-1,1}$ on $Y_{-1} \times Y_1$ which coincides with $(\cdot,\cdot)_0$ on $Y_0 \times Y_1$ and which satisfies the inequality

$$|(\zeta, y)_{-1,1}| \leq \|\zeta\|_{-1}\|y\|_1 \ , \qquad \forall\, \zeta \in Y_{-1},\ \forall\, y \in Y_1 \ . \tag{2.5}$$

With respect to the chain (2.1) we consider the three linear operators

$$A \in \mathcal{L}(Y_1, Y_{-1}) \ , \quad B \in \mathcal{L}(\mathbb{R}, Y_{-1}) \ , \quad C \in \mathcal{L}(Y_0, \mathbb{R}) \ . \tag{2.6}$$

In the control theory setting we call it *system, input* and *observation operator*, respectively. Using such a framework, certain boundary control problems for parabolic systems can be studied ([7, 10]).

Together with the operator $A \in \mathcal{L}(Y_1, Y_{-1})$ we also need the *adjoint with respect to* Y_0 *operator* $A^+ \in \mathcal{L}(Y_1, Y_{-1})$ which is given by the relation ([2])

$$(Ay, \eta)_{-1,1} = (A^+\eta, y)_{-1,1} \ , \qquad \forall\, y, \eta \in Y_1 \ . \tag{2.7}$$

If $A^+ = A$ the operator A is called *self-adjoint with respect to* Y_0. The adjointness with respect to Y_0 can be introduced similarly for linear operators acting between other spaces in the chain (2.1).

The construction of some auxilary evolutionary variational equation is based on the following function spaces which we shortly introduce.

If $-\infty \leq T_1 < T_2 \leq +\infty$ are two arbitrary numbers, we define the norm for Bochner measurable functions ([13]) in $L^2(T_1, T_2; Y_j)$, $\quad j = 1, 0, -1$, by

$$\|y\|_{2,j} := \left(\int_{T_1}^{T_2} \|y(t)\|_j^2 \ \mathrm{d}t \right)^{1/2} \ . \tag{2.8}$$

Let $\mathcal{W}(T_1, T_2)$ denote the space of functions y such that $y \in L^2(T_1, T_2; Y_1)$ and $\dot{y} \in L^2(T_1, T_2; Y_{-1})$ equipped with the norm

$$\|y\|_{\mathcal{W}(T_1,T_2)} := \left(\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1}^2 \right)^{1/2} \ . \tag{2.9}$$

By an embedding theorem ([8, 13]) one can assume that any function from $\mathcal{W}(T_1, T_2)$ belongs to $C(T_1, T_2; Y_0)$.

Throughout the paper we use the following assumptions about the operators A, B, C.

(A1) For any $T > 0$ and any $f \in L^2(0, T; Y_{-1})$ the problem

$$\dot{y} = Ay + f(t) \quad , \quad y(0) = y_0 \tag{2.10}$$

is well-posed, i.e. for arbitrary $y_0 \in Y_0, f \in L^2(0,T;Y_{-1})$ there exists a unique solution $y \in \mathcal{W}(0,T)$ satisfying (2.10) in a variational sense and depending continuously on the initial data, i.e.

$$\|y(\cdot)\|^2_{\mathcal{W}(0,T)} \le c_1\|y_0\|_0^2 + c_2\|f(\cdot)\|^2_{2,-1} \,, \tag{2.11}$$

where $c_1 > 0$ and $c_2 > 0$ are some constants.

(A2) The operator A is *Hurwitz*, i.e. any solution of

$$\dot{y} = Ay, \qquad y(0) \in Y_0 \,, \tag{2.12}$$

is exponentially decreasing for $t \to +\infty$.

(A3) The operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is *regular* ([6, 7]), i.e. for any $T > 0$, $y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0,T;Y_1)$ the solution of the direct problem (2.10) and the solution of the adjoint problem (understood in the above sense)

$$\dot{z} = -A^+ z + f(t), \qquad z(T) = z_T \,, \tag{2.13}$$

are strongly continuous in t in the norm of Y_1.

(A4) The pair (A,B) is L^2-*controllable*, i.e. for arbitrary $y_0 \in Y_0$ there exists a control $\xi(\cdot) \in L^2(0,\infty;\mathbb{R})$ such that the problem

$$\dot{y} = Ay + B\xi, \qquad y(0) = y_0 \tag{2.14}$$

is well-posed in the variational sense on $(0,+\infty)$.

Let us denote by H^c and L^c the complexification of a linear space H and a linear operator L, respectively, and introduce by

$$\chi(s) = C^c(A^c - sI^c)^{-1}B^c, \qquad s \in \rho\,(A^c) \tag{2.15}$$

the *transfer operator function* of the triple (A^c, B^c, C^c).

(A5) There exist numbers $\kappa_0 > 0$ and $\beta > 0$ such that

$$\frac{1}{\kappa_0} + \operatorname{Re}\chi(i\omega) > \beta \,, \qquad \forall\, \omega \in \mathbb{R} \,. \tag{2.16}$$

Theorem 2.1. *Assume for the linear operators* $A \in \mathcal{L}(Y_1, Y_{-1}), B \in \mathcal{L}(\mathbb{R}, Y_{-1})$ *and* $C \in \mathcal{L}(Y_0, \mathbb{R})$ *that the assumptions* **(A1)**–**(A5)** *are satisfied. Then there exists an operator* $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$*, self-adjoint and positive in* Y_0*, and a number* $\varepsilon > 0$ *such that*

$$(Ay + B\xi, Py)_{-1,1} + \xi(Cy - \xi\kappa_0^{-1}) \le -\varepsilon(\|y\|_1^2 + \xi^2), \quad \forall\,(y,\xi) \in Y_1 \times \mathbb{R} \,. \tag{2.17}$$

Proof. Consider in $Y_1 \times \mathbb{R}$ the quadratic form $F(y,\xi) = \xi(Cy - \xi\kappa_0^{-1})$ and their Hermitian extension $F^c(y,\xi) = \operatorname{Re}(\xi^* C^c y) - |\xi|^2\kappa_0^{-1}$ in $Y_1^c \times \mathbb{C}$. From the Likhtarnikov-Yakubovich theorem ([6]) it follows that under the conditions **(A1)**, **(A3)**, **(A4)** and the frequency-domain condition

$$\operatorname{Re}(\xi^* C^c y) - |\xi|^2\kappa_0^{-1} < -\beta|\xi|^2 \,, \tag{2.18}$$
$$\forall\, \xi \in \mathbb{C} \setminus \{0\} \quad \forall\, \omega \in \mathbb{R} \quad \forall\, y \in Y_1^c : i\omega y = A^c y + B^c\xi \,,$$

there exists an operator $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ self-adjoint in Y_0, such that (2.17) is satisfied. As it is easy to see, inequality (2.18) is equivalent to (2.16).

Let us show that $P \geq 0$. Introduce on Y_0 the Lyapunov functional $V(y) := (y, Py)_0$. Putting in (2.17) $\xi = 0$ we get the inequality

$$(Ay, Py)_{-1,1} \leq -\varepsilon \|y\|_1^2 , \qquad \forall\, y \in Y_1 . \tag{2.19}$$

Thus we have along an arbitrary solution $y(\cdot)$ of $\dot{y} = Ay$ with $y(0) = y_0 \in Y_0$ on an interval $[0, t]$ the inequality

$$V(y(t)) \leq V(y_0) - \varepsilon \int_0^t \|y(\tau)\|_1^2 \, \mathrm{d}\tau . \tag{2.20}$$

From **(A2)** and (2.20) it follows for $t \to +\infty$ that

$$0 \leq V(y_0) - \varepsilon \int_0^\infty \|y(\tau)\|_1^2 \, \mathrm{d}\tau .$$

But this implies that $V(y_0) > 0$ if $y_0 \neq 0$. □

The operator P from Theorem 2.1, positive and self-adjoint in Y_0, satisfies the Lyapunov inequality (2.19) and generates a Lyapunov (or energy) functional $V(y) = (y, Py)_0$. For this reason we call P *Lyapunov operator*.

In the following we suppose the properties **(A1)**–**(A5)**. Thus we can assume that there exists a Lyapunov operator P and a number $\varepsilon > 0$ satisfying (2.17). Note that the operator P can be explicitly determined as solution of a Hamiltonian system of equations ([6]). The number $\varepsilon > 0$ can be estimated with the knowledge of β. Our aim is to derive with the help of P a new Gelfand chain from (2.1) which is better adapted to the nonlinear system which will be investigated in Sec. 3.

Consider in Y_0 the new scalar product $(\cdot, \cdot)_{0,P}$ given by

$$(y, \eta)_{0,P} := (y, P\eta)_0 , \qquad \forall\, y, \eta \in Y_0 .$$

The associated norm is denoted by $\|\cdot\|_{0,P}$. The completion of Y_0 w.r.t. the scalar product $(\cdot)_{0,P}$ gives the Hilbert space $Y_{0,P}$. The space Y_1 is dense in $Y_{0,P}$ since Y_1 is dense in Y_0 and Y_0 is dense in $Y_{0,P}$. By (2.2) and the boundedness of P it follows that for all $y \in Y_1$

$$\|y\|_{0,P} = (y, Py)_0^{1/2} \leq \|P\|^{1/2} \|y\|_0 \leq \|P\|^{1/2} c_1 \|y\|_1 . \tag{2.21}$$

But this means that the inclusion $Y_1 \subset Y_{0,P}$ is continuous. Thus we can continue the inclusion $Y_1 \subset Y_{0,P}$ to a Gelfand rigged chain

$$Y_1 \subset Y_{0,P} \subset Y_{-1,P} \tag{2.22}$$

of Hilbert spaces. In order to define the negative space in this chain explicitly we introduce ([2]) on $Y_{0,P}$ the negative norm $\|\cdot\|_{-1,P}$ given on $Y_{0,P}$ by

$$\|y\|_{-1,P} := \sup_{0 \neq \eta \in Y_1} \frac{|(y, \eta)_{0,P}|}{\|\eta\|_1} . \tag{2.23}$$

The completion of $Y_{0,P}$ in this norm gives the negative space $Y_{-1,P}$ in the chain (2.22).

Let us denote the pairing between $Y_{-1,P}$ and Y_1 by $(\cdot,\cdot)_{-1,P;1}$. We extend by continuity the operators A, B and C from (2.6) to operators

$$A_P \in \mathcal{L}(Y_1, Y_{-1,P}) \ , \ B_P \in \mathcal{L}(\mathbb{R}, Y_{-1,P}) \ , \ C_P \in \mathcal{L}(Y_{0,P}, \mathbb{R}) \ . \tag{2.24}$$

Denote for $-\infty \le T_1 < T_2 \le +\infty$ by $L^2(T_1, T_2; Y_j)$ with $j = 0, P$ and $j = -1, P$ the Bochner measurable functions for which the norm $\|\cdot\|_{2,j}$, defined by (2.8), is finite. Let $\mathcal{W}_P(T_1, T_2)$ be the space of functions such that

$$y \in L^2(T_1, T_2; Y_1) \text{ and } \dot{y} \in L^2(T_1, T_2; Y_{-1,P}) \ ,$$

equipped with the norm

$$\|y\|_{\mathcal{W}_p(T_1,T_2)} := (\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1,P}^2)^{1/2} \ .$$

3. Variational inequalities with hysteresis nonlinearities related to nonstandard Gelfand riggings. Assume that there are linear operators (2.6) satisfying **(A1)–(A5)**. As it was shown in Sec. 2, under these assumptions there exists a nonstandard Gelfand rigging (2.22) and the operator extensions (2.24). These operators will play the role of the "linear part" of our evolutionary variational inequality.

Assume further that

$$\varphi : \mathcal{D}(\varphi) \subset C([0,\infty)) \times \mathbb{R} \to C([0,\infty)) \tag{3.1}$$

is a hysteresis operator which has the following properties:

(P1) The domain of definition of (3.1) is given by the set-valued function $\mathcal{E} : \mathbb{R} \to 2^{\mathbb{R}}$, i.e.

$$\mathcal{D}(\varphi) = \big\{(w, \xi_0) \in C([0,\infty)) \times \mathbb{R} \,\big|\, \xi_0 \in \mathcal{E}(w(0))\big\}.$$

The operator φ is causal, that is if $(w_i, \xi_0) \in \mathcal{D}(\varphi), i = 1, 2$, and $w_1 \equiv w_2$ on $[0, t)$ then $\varphi(w_1, \xi_0)(t) = \varphi(w_2, \xi_0)(t)$.

(P2) For any $T > 0$ the operator φ is regarded as map from $\{(w_{|[0,T]}, \xi_0) \,|\, (w, \xi_0) \in \mathcal{D}(\varphi)\}$ into $C([0, T])$ satisfying with the constant $\kappa_0 > 0$ from Theorem 2.1 the inequality

$$\|\varphi(w_1, \xi_0) - \varphi(w_2, \xi_0)\,\|_{C([0,T])} \le \kappa_0 \|w_1 - w_2\|_{C([0,T])}$$

for any $(w_i, \xi_0) \in \mathcal{D}(\varphi), i = 1, 2$.
For any $(w, \xi_0) \in \mathcal{D}(\varphi)$ with $w \in W^{1,1}(0, T)$ the function $\xi(t) \equiv \varphi(w, \xi_0)(t)$ belongs also to $W^{1,1}(0, T)$ and satisfies

$$0 \le \dot{\xi}(t)\dot{w}(t) \le \kappa_0 \dot{w}(t)^2 \tag{3.2}$$

for a.a. $t \ge 0$.

(P3) φ is *limit-continuous* ([14]), i.e. any $w \in W^{1,2}(0,\infty;\mathbb{R})$ with $w(t) \to w_\infty$ and $\varphi(w, \xi_0)(t) \to \xi_\infty$ for $t \to \infty$ implies that $\xi_\infty \in \mathcal{E}(w_\infty)$ and $\varphi(w_\infty, \xi_\infty)(t) \equiv \xi_\infty$.

Let $\psi : Y_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower-semicontinuous function with domain $\mathcal{D}(\psi) \neq \emptyset$. We suppose that ψ is continuous on $\overline{\mathcal{D}(\psi)}^{Y_{0,P}}$ (the closure in $Y_{0,P}$). For any $y \in Y_1$ the *subdifferential* $\partial\psi_P(y)$ *of* ψ *at* y *with respect to the pairing* $(\cdot,\cdot)_{-1,P;1}$ between $Y_{-1,P}$ and Y_1 is the set of all $\zeta \in Y_{-1,P}$ such that

$$\psi(y) \le \psi(\eta) + (\zeta, y - \eta)_{-1,P;1} \ , \quad \forall\, \eta \in Y_1 \ . \tag{3.3}$$

It follows immediately from (3.3) that $\partial\psi_P$ is a monotone (possibly multivalued) operator from Y_1 to $Y_{-1,P}$ with respect to the pairing $(\cdot,\cdot)_{-1,P;1}$. It is easy to see that in case when $P = P^* : Y_0 \to Y_0$ is invertible, the subdifferential $\partial_P\psi(y)$ with respect to the pairing $(\cdot,\cdot)_{-1,P;1}$ is given by $\partial_P\psi(y) = P^{-1}\partial\psi(y)$, where $\partial\psi(y)$ is the usual subdifferential of ψ at y.

Let us introduce the following property for ψ.

(P4) For any $y \in Y_1$ the subdifferential $\partial_P\psi(y)$ at y with respect to the pairing $(\cdot,\cdot)_{-1,P;1}$ is maximal monoton.

Note that in the case when $P = I$ condition **(P4)** can be expressed by the convexity and lower-semicontinuity of ψ. Consider on an arbitrary interval $(0,T)$ the evolutionary variational inequality

$$\begin{aligned}
&(\dot y - A_P y - B_P\xi(t), \eta - y)_{-1,P;1} + \psi(\eta) - \psi(y) \ge 0\,,\\
&\xi(t) = \varphi(w,\xi_0)(t)\,, \quad w(t) = C_P y(t)\,,\\
&y(0) = y_0 \in \overline{\mathcal{D}(\psi)}^{Y_{0,P}}, \xi_0 \in \mathcal{E}(w(0))\,, \quad \forall\, \eta \in Y_1\,,\ \text{a.a. } t \in (0,T)\,.
\end{aligned} \tag{3.4}$$

A pair of functions $\{y,\xi\} \in \mathcal{W}_P(0,T) \cap C(0,T;Y_{0,P}) \times W^{1,2}(0,T;\mathbb{R})$ which satisfies (3.4) is called *solution* of the variational inequality on $(0,T)$ with initial state $\{y_0,\xi_0\}$.

Note that (3.4) can be understood as generalization of the following standard variational inequality.

Consider with respect to the chain (2.1), the linear operators (2.6), the hysteresis operator φ, and the function ψ the variational inequality

$$\begin{aligned}
&(\dot y - Ay - B\xi(t), \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \ge 0\,,\\
&\xi(t) = \varphi(w,\xi_0)(t)\,, \quad w(t) = Cy(t)\,,\\
&y(0) = y_0 \in \overline{\mathcal{D}(\psi)}^{Y_0}\,, \quad \xi_0 \in \mathcal{E}(w(0))\,, \quad \forall\, \eta \in Y_1\,,\ \text{a.a. } t \in (0,T)\,.
\end{aligned} \tag{3.5}$$

A solution of (3.5) is defined as for (3.4) with $P = I$. It is clear that (3.4) for $P = I$ goes over into (3.5). In many cases however (3.4) has for certain $P \in \mathcal{L}(Y_0,Y_{-1}) \cap \mathcal{L}(Y_1,Y_0)$ a solution, but (3.5) doesn't have a solution.

In the following we have to suppose some regularity of the solutions of (3.4):

(P5) For each $T > 0, y(0) \in \overline{\mathcal{D}(\psi)}^{Y_{0,P}}$ and $\xi(0) \in \mathcal{E}(C_P y(0))$ there exists exactly one solution $\{y,\xi\}$ of (3.4) satisfying

$$\{y,\xi\} \in W^{1,2}(0,T;Y_1) \times W^{1,2}(0,T;\mathbb{R})\,. \tag{3.6}$$

4. Frequency-domain conditions for exponential stability of stationary solutions. Let us assume that all the assumptions of Section 2 and Section 3 are satisfied. This means that we can consider the inequality (3.4) with respect to the non-standard chain (2.22). Assume also that the set of stationary solutions (equilibria) of (3.4) is non-empty. Recall that a pair $\{y_\infty,\xi_\infty\} \in \overline{\mathcal{D}(\psi)}^{Y_{0,P}} \times \mathbb{R}$ with $\xi_\infty \in \mathcal{E}(C_P y_\infty)$ is a stationary solution of (3.4) if and only if this pair satisfies the (stationary) inequality

$$(-A_P y_\infty - B_P\xi_\infty, \eta - y_\infty)_{-1,P;1} + \psi(\eta) - \psi(y_\infty) \ge 0\,, \quad \forall\, \eta \in Y_1\,. \tag{4.1}$$

Let us state now the main result of this paper.

In the second part of the statement we need the following supplementary assumptions.

(P6) There exists a constant $\kappa_1 \in (0, 1/6)$ such that

$$|\psi(y)| \le \kappa_1 \| y\|_{0,P} , \quad \forall\, y \in \mathcal{D}(\psi) .$$

(P7) For the constants $\varepsilon > 0$ and $\kappa_0 > 0$ from (2.17) and (3.2), respectively, we have $\|C_P\| < 2(\varepsilon - \kappa_0^{-1})$, where $\|C_P\|$ is the operator norm of $C_P \in \mathcal{L}(Y_{0,P}, \mathbb{R})$.

Theorem 4.1. *Suppose that $\{y(\cdot), \xi(\cdot)\}$ is an arbitrary solution of* (3.4) *on* $(0,\infty)$ *satisfying* **(P5)**. *Then at every $t > 0$ the right derivative $D^+y(t)$ exists and there is a number $\lambda > 0$, independent on the concrete solution, such that for every $t \ge s > 0$*

$$\|D^+y(t)\|_{0,P}^2 \le e^{-2\lambda(t-s)} \|D^+y(s)\|_{0,P} . \tag{4.2}$$

Hence $\|\dot y\|_{0,P}$ and $\dot\xi$ are in $L^1(0,\infty), \dot y \in L^1(0,\infty; Y_{0,P})$ and there are stationary solutions $\{y_\infty, \xi_\infty\}$ of (3.4) *such that*

$$y(t) \to y_\infty \text{ in } Y_{0,P} \quad \text{and} \quad \xi(t) \to \xi_\infty \tag{4.3}$$

as $t \to +\infty$. Assume additionally the properties **(P6)** *and* **(P7)**. *Then the convergence* (4.3) *is exponentially, i.e. there exist constants $c_i > 0$, $i = 1, 2, 3, 4$, independently on $\{y_0, \xi_0\}$, such that*

$$\|y(t) - y_\infty\|_{0,P}^2 \le e^{-2\lambda(t-s)} \left[c_1 \|y(s)\|_{0,P}^2 + c_2\, \xi(s)^2\right] \tag{4.4}$$

and

$$(\xi(t) - \xi_\infty)^2 \le e^{-2\lambda(t-s)} \left[c_3 \|y(s)\|_{0,P}^2 + c_4\, \xi(s)^2\right] \tag{4.5}$$

for all $t \ge s > 0$.

Proof. If we insert in (3.4) $\eta = y(t+h)$ with some $h > 0$ we get for a.e. $t \ge 0$

$$\begin{aligned}\Big(\dot y(t) - A_P y(t) - B_P\, \xi(t), y(t+h) - y(t)\Big)_{-1,P;1}& \\ + \psi(y(t+h)) - \psi(y(t)) \ge 0 .&\end{aligned} \tag{4.6}$$

Now we put in (3.4) $t = t + h$, $\eta = y(t)$ and receive for a.e. $t \ge 0$

$$\begin{aligned}\Big(\dot y(t+h) - A_P y(t+h) - B_P \xi(t+h), y(t) - y(t+h))_{-1,P;1}& \\ + \psi(y(t)) - \psi(y(t+h))\Big) \ge 0 .&\end{aligned} \tag{4.7}$$

The addition of (4.6) and (4.7) gives for a.e. $t \ge 0$ the inequality

$$\begin{aligned}\Big(\dot y(t+h) - y(t) - A_P\big[y(t+h) - y(t)\big] - B_P\big[\xi(t+h) - \xi(t)\big],& \\ y(t+h) - y(t)\Big)_{-1,P;1} \le 0 .&\end{aligned} \tag{4.8}$$

It follows that for a.e. $t \ge 0$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|y(t+h) - y(t)\|_{0,P}^2 \\ &- \Big(A_P\big[y(t+h) - y(t)\big] + B_P\big[\xi(t+h) - \xi(t)\big], \big[y(t+h) - y(t)\big]\Big)_{-1,P;1} \le 0 .\end{aligned} \tag{4.9}$$

If we divide (4.9) by h^2 we get for a.e. $t \geq 0$

$$\begin{aligned} &\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{y(t+h)-y(t)}{h}\right\|^2_{0,P} \\ &-\left(A_P\left[\frac{y(t+h)-y(t)}{h}\right]+B_P\left[\frac{\xi(t+h)-\xi(t)}{h}\right],\frac{y(t+h)-y(t)}{h}\right)_{-1,P;1}\leq 0\,. \end{aligned} \tag{4.10}$$

Since $\xi(t) = \varphi(C_P y, \xi(0))(t)$ belongs to $W^{1,2}(0,T)$ for each $T > 0$, the solution $y(t)$ is right differentiable at any interval $[s,t] \subset (0,T]$ (see [1]). Thus, letting $h \downarrow 0$ and integrating (4.10) over $[s,t]$, it follows that

$$\|D^+y(t)\|^2_{0,P} - \|D^+y(s)\|^2_{0,P} \leq 2\int_s^t (A_P\dot{y}(\tau) + B_P\dot{\xi}(\tau), \dot{y}(\tau))_{-1,P;1}\,\mathrm{d}\tau. \tag{4.11}$$

Since $\dot{y}(t) \in Y_1$ and $P \in \mathcal{L}(Y_0, Y_1)$ the last inequality can be written as

$$\|D^+y(t)\|^2_{0,P} - \|D^+y(s)\|^2_{0,P} \leq 2\int_s^t (A\dot{y}(\tau) + B\dot{\xi}(\tau), P\dot{y}(\tau))_0\,\mathrm{d}\tau\,. \tag{4.12}$$

From (2.17) and (3.2) it follows that for a.e. $t \geq 0$

$$-\left(A\dot{y}(t) + B\dot{\xi}(t),\, P\dot{y}(t)\right)_0 \geq \varepsilon\|y(t)\|_1^2\,. \tag{4.13}$$

Since P is bounded there exists a constant $M > 0$ such that

$$\frac{\varepsilon}{M}(\eta, P\eta)_0 \leq \varepsilon\|\eta\|_1^2\,, \quad \forall\,\eta \in Y_1\,. \tag{4.14}$$

It follows from (4.12)–(4.14) with $\lambda := \frac{\varepsilon}{M}$ that

$$\|D^+y(t)\|^2_{0,P} - \|D^+y(s)\|^2_{0,P} + 2\lambda\int_s^t (\dot{y}(\tau), P\dot{y}(t))_0\,\leq 0\,. \tag{4.15}$$

Since $\dot{y}(t) = D^+y(t)$ a.e. on $[s,t]$ we conclude from (4.15) that (4.2) is true.

From (4.15) it follows that $(\dot{y}(t), P\dot{y}(t))_0^{1/2} \in L^1(0,\infty)$. From the convergence of the integral $\int_0^\infty (\dot{y}, P\dot{y})_0^{1/2}\,\mathrm{d}t$ we get the convergence in $Y_{0,P}$ of

$$y_\infty = \lim_{t\to\infty} y(t) = y(0) + \int_0^\infty \dot{y}(t)\,\mathrm{d}t\,. \tag{4.16}$$

Now we conclude that $w(t) = C_P y(t) \to w_\infty$ for $t \to \infty$ and $\xi(t) = \varphi(w, \xi_0)(t) \to \xi_\infty$ as $t \to \infty$, where $w_\infty, \xi_\infty \in \mathbb{R}$. From the limit continuity of φ (property **(P3)**) we see that $\xi_\infty \in \mathcal{E}(w_\infty)$ and $\varphi(w_\infty, \xi_\infty)(t) \equiv \xi_\infty$. The continuity of ψ on $\overline{\mathcal{D}(\psi)}^{Y_{0,P}}$ implies that $\psi(y(t)) \to \psi(y_\infty)$ as $t \to \infty$. This and (4.1) show that for each $\eta \in Y_1$

$$(-A_P y_\infty - B_P\xi_\infty, \eta - y_\infty)_{-1,P;1} + \psi(\eta) - \psi(y_\infty) \geq 0\,. \tag{4.17}$$

Thus $\{y_\infty, \xi_\infty\}$ is a stationary solution of (3.4). Equation (4.16), inequality (4.2) and the representation $y(t) = \int_0^\infty \dot{y}(\tau)\,\mathrm{d}\tau + y(0)$ give for each $t > 0$ the estimate

$$\|\,y(t) - y_\infty\|^2_{0,P} \leq \int_t^\infty \|\,\dot{y}(\tau)\|_{0,P}\,\mathrm{d}\tau \leq \frac{1}{2\lambda}e^{-2\lambda(t-s)}\|\,D^+y(s)\|^2_{0,P}\,. \tag{4.18}$$

The next step is to estimate $\| D^+y(s)\|^2_{0,P}$ with the help of $\| y(s) \|^2_{0,P}$ and $\xi(s)^2$. If we put in (3.4) $t = 0$ and $\eta = -D^+y(s) + y(s)$ we receive

$$(D^+y(s) - A_P y(s) - B_P\xi(s), D^+y(s))_{-1,P;1} \\ -\psi\left(-D^+y(s) + y(s)\right) + \psi(y(s)) \leq 0 \; . \tag{4.19}$$

From (4.19) and **(P6)** it follows that

$$\| D^+y(s)\|^2_{0,P} \leq \tilde{c}_5 \| y(s)\|^2_{0,P} + \tilde{c}_6 \xi(s)^2 + \left(\tfrac{1}{2} + 3\kappa_1\right) \| D^+y(s)\|^2_{0,P} \tag{4.20}$$

with $\tilde{c}_5 = \frac{1}{2}\| A_P\|^2 + \frac{1}{2}\| A_P\| \, \|B_P\| + 3\kappa_1$ and $\tilde{c}_6 = \frac{1}{2}\big(\| A_P\| \, \| B_P\| + \| B_P\|^2 \,\big)$. Here $\|A_P\|$ and $\|B_P\|$ denote the operator norms of $A_P \in \mathcal{L}(Y_1, Y_{-1,P})$ and $B_P \in \mathcal{L}(\mathbb{R}, Y_{-1,P})$, respectively.

Using again **(P6)** we get from (4.20) the inequality

$$\| D^+y(s)\|^2_{0,P} \leq c_5 \|y(s)\|^2_{0,P} + c_6 \, \xi(s)^2 \tag{4.21}$$

with $c_5 = \dfrac{\tilde{c}_5}{\frac{1}{2} - 3\kappa_1}$ and $c_6 = \dfrac{\tilde{c}_6}{\frac{1}{2} - 3\kappa_1}$. In oder to derive inequality (4.5) we put $\eta = y_\infty$ into (3.4) and receive for a.e. $t \geq 0$

$$\big(\dot{y}(t) - A_P y(t) - B_P\xi(t), y_\infty - y(t)\big)_{-1,P;1} + \psi(y_\infty) - \psi(y(t)) \geq 0 \; . \tag{4.22}$$

If we put $\eta = y(t)$ into inequality (4.1) we obtain

$$(-A_P y_\infty - B_P\xi_\infty, y(t) - y_\infty)_{-1,P;1} + \psi(y(t)) - \psi(y_\infty) \geq 0 \; . \tag{4.23}$$

If we add (4.22) and (4.23), we receive for a.e. $t \geq 0$ the inequality

$$\big(- \dot{y} + A_P[y(t) - y_\infty] + B_P[\xi(t) - \xi_\infty] \, , \, y(t) - y_\infty)_{-1,P;1} \geq 0 \; . \tag{4.24}$$

By assumption $y(t) - y_\infty \in Y_1$. Thus we can use (2.17) to get the inequality

$$\begin{aligned} &\big(A_P[y(t) - y_\infty] + B_P[\xi(t) - \xi_\infty], y(t) - y_\infty)_{-1,P;1} \\ &\leq - \varepsilon\big[\| y(t) - y_\infty\|^2_1 + (\xi(t) - \xi_\infty)^2\big] - \big(\xi(t) - \xi_\infty)(C_P(y(t) - y_\infty) \\ &\qquad - (\xi(t) - \xi_\infty)\kappa_0^{-1}\big) \; . \end{aligned} \tag{4.25}$$

It follows from (4.24) and (4.25) that

$$(\varepsilon - \kappa_0^{-1})(\xi(t) - \xi_\infty)^2 \leq |\xi(t) - \xi_\infty| \, \|C_P\| \, \| y(t) - y_\infty\|_{0,P} \\ + \frac{1}{2} \left\| D^+y(t)\right\|^2_{0,P} + \frac{1}{2} \left\| y(t) - y_\infty\right\|^2_{0,P} \; . \tag{4.26}$$

If we use in (4.26) inequalities (4.2), (4.21) and the property **(P7)** we get immediately inequality (4.5) with

$$c_3 = \frac{\frac{1}{2}\, c_1(\|C_P\| + 1) + \frac{1}{2} c_5}{\varepsilon - \kappa_0^{-1} - \frac{1}{2}\|C_P\|} \qquad \text{and}$$

$$c_4 = \frac{\frac{1}{2}\, c_2(\|C_P\| + 1) + \frac{1}{2} c_6}{\varepsilon - \kappa_0^{-1} - \frac{1}{2}\|C_P\|} \; .$$

□

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