Jan Prüss; Jürgen Saal; Gieri Simonett
Analytic solutions for the classical two-phase Stefan problem


Persistent URL: http://dml.cz/dmlcz/700438

Terms of use:
© Comenius University, 2007

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library
http://project.dml.cz
ANALYTIC SOLUTIONS FOR THE CLASSICAL
TWO-PHASE STEFAN PROBLEM∗
JAN PRÜSS, JÜRGEN SAAL†, AND GIERI SIMONETT

Abstract. A survey of the results obtained in [22] is presented. In [22] the authors prove the
existence of a local-in-time solution for the classical two-phase Stefan problem that is analytic in space
and time. The result is based on $L_p$ maximal regularity, which is proved first, and the implicit function
theorem.

Key words. Classical Stefan problem, free boundary problem, phase transition, maximal regularity,
analytic solutions.


1. Introduction and main result. Consider a family $\Gamma = \{\Gamma(t) : t \geq 0\}$ of
hypersurfaces in $\mathbb{R}^{n+1}$, where each individual hypersurface is assumed to be a graph over
$\mathbb{R}^n$, that is, $\Gamma(t) = \text{graph}(\rho(t))$ for some $\rho(t) : \mathbb{R}^n \to \mathbb{R}$. Moreover, let $\Omega^+(t)$ and $\Omega^-(t)$
denote the domain above and below $\Gamma(t)$, respectively, that is,
$$\Omega^\pm(t) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : \pm y > \pm \rho(t, x)\}.$$ We set $\Omega(t) := \Omega^+(t) \cup \Omega^-(t)$ and consider the following problem: Given $\Gamma_0 = \text{graph}(\rho_0)$
and $u_0 : \Omega(0) \to \mathbb{R}$, determine a family $\Gamma = \{\Gamma(t) : t \geq 0\}$ and a function $u : \bigcup_{t \geq 0}(\{t\} \times \Omega(t)) \to \mathbb{R}$ such that

$$\begin{cases}
(\partial_t - c\Delta)u = 0 & \text{in } \bigcup_{t>0}(\{t\} \times \Omega(t)), \\
\gamma u = 0 & \text{on } \bigcup_{t>0}(\{t\} \times \Gamma(t)), \\
v = -[c\partial_\nu u] & \text{on } \bigcup_{t>0}(\{t\} \times \Gamma(t)), \\
u(0) = u_0 & \text{in } \Omega(0), \\
\Gamma(0) = \Gamma_0,
\end{cases}
$$ (1.1)

where $\gamma$ stands for the trace operator, $V$ denotes the normal velocity of $\Gamma$, and $\nu$ is the
unit normal vector, pointing into $\Omega^+(t)$ (1). Given any function $v : \Omega(t) \to \mathbb{R}$, we write $v^+$
and $v^-$ for the restriction of $v$ to $\Omega^+(t)$ and $\Omega^-(t)$, respectively. Moreover, we admit the
possibility of two different diffusion coefficients in $\Omega^\pm(t)$, i.e. $c$ is given as

$$c(t, x, y) = \begin{cases}
c_+, & (x, y) \in \Omega^+(t), \\
c_-, & (x, y) \in \Omega^-(t),
\end{cases}$$ (1.2)

where $c_+, c_-$ are strictly positive constants. Using this notation, let $[c\partial_\nu u]$ denote the
jump of the normal derivatives of $u$ across $\Gamma(t)$, that is,

$$[c\partial_\nu u] := c_+ \gamma \partial_\nu u^+ - c_- \gamma \partial_\nu u^-.$$
Of course, $u_0$ is a given initial value for $u$ and $\Gamma_0$ describes the initial position of $\Gamma$.

Problem (1.1) is called the classical two-phase Stefan problem which is a model for phase transitions in liquid-solid systems and accounts for heat diffusion and exchange of latent heat in a homogeneous medium. In a typical physical situation the domain $\Omega$ is occupied by a liquid and a solid phase, say water and ice, that are separated by the interface $\Gamma$. Due to melting or freezing, the corresponding regions occupied by water and ice will change and, consequently, the interface $\Gamma$ will also change its position and shape, which leads to the free boundary problem (1.1).

In the classical Stefan problem one assumes that the temperatures $u^+$ and $u^-$ coincide at the interface $\Gamma$ (where the two phases are in contact), that is, one requires

$$u^+ = u^- = 0 \quad \text{on} \quad \Gamma,$$

where $0$ is the melting temperature.

The Stefan problem has been studied in the mathematical literature for over a century, see [23, 20] and [26, pp. 117–120] for a historic account, and has attracted the attention of many prominent mathematicians, see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24].

To formulate our main result, let $W^s_p(\mathbb{R}^n)$, $s \geq 0$, $p \in (1, \infty)$, denote the Sobolev-Slobodeckij spaces, cf. [25]. Then we have

**Theorem 1.1.** Let $p > n + 3$. Then there is a number $\eta > 0$ such that the following holds: Given $(u_0, \rho_0) \in W^{2-2/p}(\Omega(0)) \times W^{2-2/p}(\mathbb{R}^n)$ with

$$\gamma u_0^+ = 0, \quad \pm u_0^+ > 0 \quad \text{on} \quad \Omega^\pm(0), \quad \alpha_\pm := \partial_{\nu} u_0^\pm(0, \rho_0(0)) > 0,$$

and

$$\|\rho_0\|_{BUC^1(\mathbb{R}^n)} + \|\partial_{\nu} u_0^\pm - \alpha_\pm\|_{BUC(\Gamma_0)} \leq \eta,$$

there exists $T = T(u_0, \rho_0)$ and a unique solution $(u, \Gamma)$, where $\Gamma(t) = \text{graph}(\rho(t))$, for the Stefan problem (1.1) that is analytic in space and time. More precisely, we have that

$$M = \bigcup_{t \in (0, T)} \{\{t\} \times \Gamma(t)\} \quad \text{is a real analytic manifold}$$

and that $u^\pm \in C^\omega(\overline{\Omega}^\pm_T, \mathbb{R})$, with

$$\overline{\Omega}^\pm_T := \{(t, (x, y)) \in (0, T) \times \mathbb{R}^{n+1} : (x, y) \in \Omega^\pm(t)\}.$$
into a problem in $\mathbb{R}^{n+1}$. For this purpose we define

$$
\Theta : (0, T) \times \mathbb{R}^{n+1} \to Q_T := \bigcup_{t \in (0, T)} \{t\} \times \Omega(t),
$$

$$
\Theta(t, x, y) := (t, x, y + \rho_E(t, x, y)),
$$

where $\Gamma(t) = \text{graph}(\rho(t))$ as defined in the last section and $\rho_E$ is a suitable extension of $\rho$ to $(0, T) \times \mathbb{R}^{n+1}$. We denote by

$$
u = \Theta^* v = v \circ \Theta^{-1} \quad \text{and} \quad v = \Theta^* u = u \circ \Theta$$

the push-forward and pull-back respectively, and set $J = (0, T)$. Then it can be shown that (1.1) is formally equivalent to the system

\begin{align*}
\partial_t v - c \Delta v &= F(v, \rho_E), \quad \text{in } J \times \mathbb{R}^{n+1}, \\
\gamma v^+ &= 0, \quad \text{on } J \times \mathbb{R}^n, \\
\partial_t \rho + [c_\gamma \partial_y (v - a \rho_E)] &= H(v, \rho_E), \quad \text{on } J \times \mathbb{R}^n, \\
v(0) &= v_0, \quad \text{in } \mathbb{R}^{n+1}, \\
\rho(0) &= \rho_0, \quad \text{in } \mathbb{R}^n,
\end{align*}

and we will require that the function $\rho_E$ satisfies the equation

\begin{align*}
(\partial_t - c \Delta) \rho_E &= 0 \quad \text{in } J \times \mathbb{R}^{n+1}, \\
\gamma \rho_E^+ &= \rho \quad \text{on } J \times \mathbb{R}^n, \\
\rho_E(0) &= e^{-|y|(1-\Delta_x)^{\frac{1}{2}}} \rho_0 \quad \text{in } \mathbb{R}^{n+1}.
\end{align*}

Then the nonlinearities $F$ and $H$ are given by

\begin{align*}
F(v, \rho_E) &= c \left(1 + \frac{1}{1 + \frac{1}{\partial_y \rho_E}} \right) \partial^2_y v - c \frac{2 \langle \nabla_x \rho_E \nabla_y v \rangle}{1 + \partial_y \rho_E} \\
&\quad - c \left(1 + \frac{1}{1 + \frac{1}{\partial_y \rho_E}} \right) \partial^2_y \rho_E - \frac{2 \langle \nabla_x \rho_E \nabla_y \rho_E \rangle}{1 + \partial_y \rho_E} \partial_y v
\end{align*}

and

\begin{align*}
H(v, \rho_E) &= H_+(v, \rho_E) - H_-(v, \rho_E)
\end{align*}

with

\begin{align*}
H_\pm(v, \rho_E) &= c_\pm \left[ \left(1 - 1 + \frac{1}{1 + \frac{1}{\partial_y \rho_E^\pm}} \right) \gamma \partial_y v^\pm - a_\pm \gamma \partial_y \rho_E^\pm \right].
\end{align*}

Here $\langle \cdot | \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$, and $\nabla_x$ the gradient with respect to $x$. Furthermore, observe that in slight abuse of notation we also denote the pull-back $\Theta^* c$ of the diffusion coefficient $c$ introduced in (1.2) by $c$, that is, we set

\begin{align*}
c(x, y) &= \begin{cases} 
  c_+, & y > 0, \\
  c_-, & y < 0,
\end{cases}
\end{align*}
whereas
\[ a(x, y) = \begin{cases} a_+, & y > 0, \\ a_-, & y < 0, \end{cases} \quad \text{with} \quad a_\pm := \frac{\partial_y v_0^\pm(0, 0)}{1 + \partial_y \rho_E(0, 0, 0)}. \tag{2.6} \]
Observe that \( \alpha_\pm > 0 \), given in (1.4), implies that also
\[ a_\pm = \frac{\partial_y v_0^\pm(0, 0)}{1 + \partial_y \rho_E(0, 0, 0)} = (1 + |\nabla_x \rho_0(0)|^2)^{-1/2} \partial_y u_0^\pm(0, \rho_0(0)) > 0. \]
If \( F \) and \( H \) are replaced by functions belonging to suitable function spaces, then system (2.1)–(2.2) represents the linearization admitting maximal regularity as will be proved in the next section. Note that the additional term \('a \rho_E'\) appearing in the linearization of the Stefan condition
\[ [c_\gamma \partial_y (v - a \rho_E)] = c_\gamma \partial_y (v - a \rho_E)^+ - c_\gamma \partial_y (v - a \rho_E)^- = c_\gamma \partial_y (v^+ - a_\rho_E^+) - c_\gamma \partial_y (v^- - a_\rho_E^-), \]
is necessary in order to get sufficient regularity for the function \( \rho \).

3. Maximal regularity for the linearized problem. First let us introduce suitable function spaces. Let \( \Omega \subseteq \mathbb{R}^n \) be open and \( X \) be an arbitrary Banach space. By \( L_p(\Omega; X) \) and \( H^s_p(\Omega; X) \), for \( 1 \leq p \leq \infty \), \( s \in \mathbb{R} \), we denote the \( X \)-valued Lebesgue space and the Bessel potential space of order \( s \), respectively. We will also frequently make use of the Sobolev-Slobodeckij spaces \( W^{s,p}_p(\Omega; X) \), \( 1 < p < \infty \), \( s \in \mathbb{R} \setminus \mathbb{Z} \), with norm
\[ \|g\|_{W^{s,p}_p(\Omega; X)} = \|g\|_{H^{s,p}_p(\Omega; X)} + \left( \int_\Omega \int_\Omega \frac{\|g(x) - g(y)\|_X^p}{|x-y|^{n+(s-[s])p}} \, dx \, dy \right)^{1/p}, \]
where \([s]\) denotes the largest integer smaller than \( s \). Let \( T \in (0, \infty) \) and \( J = (0, T) \). We set
\[ 0W^{s}_p(J; X) := \{ u \in W^{s}_p(J; X) : u(0) = u'(0) = \ldots = u^{(k)}(0) = 0 \}, \]
\[ W^{s}_p(J; X) := \begin{cases} \{ \|g\|_{W^{s,p}_p(\Omega; X)} = \|g\|_{H^{s,p}_p(\Omega; X)} + \left( \int_\Omega \int_\Omega \frac{\|g(x) - g(y)\|_X^p}{|x-y|^{n+(s-[s])p}} \, dx \, dy \right)^{1/p}, & \text{if} \ k + \frac{1}{p} < s < k + 1 + \frac{1}{p}, \ k \in \mathbb{N} \cup \{0\}, \\ W^{s}_p(J; X), & \text{if} \ s < \frac{1}{p}. \end{cases} \]
The spaces \( 0W^{s}_p(J; X) \) are defined analogously.
In this section we consider the linearized two-phase problem
\[ \begin{align*}
(\partial_t - c\Delta) v &= f & \text{in} & \mathbb{R}^{n+1}, \\
\gamma v &= 0 & \text{on} & \mathbb{R}^n, \\
\partial_t \rho + [c_\gamma \partial_y (v - a \rho_E)] &= h & \text{on} & \mathbb{R}^n, \\
v(0) &= v_0 & \text{in} & \mathbb{R}^{n+1}, \\
\rho(0) &= \rho_0 & \text{in} & \mathbb{R}^n,
\end{align*} \tag{3.1} \]
with \( c, a \) as defined in (2.5) and (2.6). In the following, we will always assume that the function \( \rho_E \) satisfies the equation
\[ \begin{align*}
(\partial_t - c\Delta) \rho_E &= 0 & \text{in} & \mathbb{R}^{n+1}, \\
\gamma \rho_E^\pm &= \rho & \text{on} & \mathbb{R}^n, \\
\rho_E(0) &= e^{-|y|^{(1-\Delta_\gamma)^*}} \rho_0 & \text{in} & \mathbb{R}^{n+1}.
\end{align*} \tag{3.2} \]
Remarks
(a) (3.1)–(3.2) constitutes a coupled system of equations, with the functions \((v, \rho, \rho_E)\) to be determined. We will in the sequel often just refer to a solution \((v, \rho)\) of (3.1) with the understanding that the function \(\rho_E\) also has to be determined.
(b) Suppose \(\rho \in W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n))\) and \(\rho_0 \in W_p^{2-2/p}(\mathbb{R}^n)\) is given. Then the diffusion equation (3.2) admits a unique solution

\[ \rho_E \in H_p^1(J; L^p(\mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}^{n+1})). \]

(c) The solution \(\rho_E(t, \cdot)\) of equation (3.2) provides an extension of \(\rho(t, \cdot)\) to \(\mathbb{R}^{n+1}\).

We should remark that there are many possibilities to define such an extension. The chosen one is the most convenient one for our purposes.

The main result in this section is

**Theorem 3.1.** Let \(3 < p < \infty\), \(T \in (0, \infty)\), \(J = (0, T)\).

(i) There exists a unique solution \((v, \rho, \rho_E)\) to (3.1)–(3.2) with

\[
\begin{align*}
v &\in H_p^1(J; L^p(\mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}^{n+1})) \\
\rho &\in W_p^{3/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{1-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)), \\
\rho_E &\in H_p^1(J; L^p(\mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}^{n+1}))
\end{align*}
\]

if and only if the data satisfy

\[
\begin{align*}
(a) &\ f \in L_p(J; L_p(\mathbb{R}^{n+1})), \\
(b) &\ h \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)), \\
(c) &\ v_0 \in W_p^{2-2/p}(\mathbb{R}^{n+1}), \\
(d) &\ \rho_0 \in W_p^{2-2/p}(\mathbb{R}^n), \\
(e) &\ \gamma v_0 = 0.
\end{align*}
\]

(ii) If \((h(0), v_0, \rho_0) = (0, 0, 0)\), then the norm of the solution operator

\[ S_T : (f, h) \mapsto (v, \rho, \rho_E) \]

is independent of the length of \(J = (0, T)\) for any \(T \leq T_0\), with \(T_0\) arbitrary, but fixed.

We split the outline of the proof of this result in several steps, and remark that the 'only if' part follows from the trace results in [8, Section 5].

(i) In the first step we reduce the problem to the case \((h(0), v_0, \rho_0) = (0, 0, 0)\).

In fact, this can be done by constructing suitable extensions \(u_1\) and \(\eta\) in the regularity classes of \(v\) and \(\rho\) such that \(u_1(0) = v_0\) and

\[
(\eta(0), \partial_t \eta(0)) := \left(\rho_0, h(0) - [c\gamma \partial_y (v_0 - ae^{-\frac{1}{2}} \rho_0)]\right)
\]

(for the existence see [22]). Then, if \(\eta_E\) is the solution of (3.2), with \(\rho\) replaced by \(\eta\), it follows that

\[ (v - u_1, \rho - \eta, \rho_E - \eta_E) \]

\(^1\text{Actually, } u_1 \text{ is chosen as the solution of the homogeneous heat equation in } \mathbb{R}^{n+1} \text{ with initial value } v_0.\)
solves (3.1) and (3.2) with right hand sides \((f, g, 0, 0)\) and \((0, \rho, 0)\) respectively, in the right classes and such that \(g(0) = 0\).

(ii) It is also not difficult to see that by the shift \(u \mapsto e^{-t}u\) and by the use of a suitable extension operator \(E\) that maps functions \(f : J \to X\) into functions \(Ef : \mathbb{R}_+ \to X\), it suffices to consider the problems

\[
\begin{aligned}
(\partial_t + 1 - c\Delta)u &= f \; \text{ in } (0, \infty) \times \mathbb{R}^{n+1}, \\
\gamma u^+ &= 0 \; \text{ on } (0, \infty) \times \mathbb{R}^n, \\
(\partial_t + 1)\rho + [c\gamma \partial_y (u - a\rho_E)] &= h \; \text{ on } (0, \infty) \times \mathbb{R}^n, \\
u(0) &= 0 \; \text{ in } \mathbb{R}^{n+1}, \\
\rho(0) &= 0 \; \text{ in } \mathbb{R}^n,
\end{aligned}
\]

(3.4)

and

\[
\begin{aligned}
(\partial_t + 1 - c\Delta)\rho_E &= 0 \; \text{ in } (0, \infty) \times \mathbb{R}^{n+1} \\
\gamma \rho_E^+ &= \rho \; \text{ on } (0, \infty) \times \mathbb{R}^n, \\
\rho_E(0) &= 0 \; \text{ in } \mathbb{R}^{n+1}.
\end{aligned}
\]

(3.5)

Applying the Fourier-Laplace transform in \((t, x)\), denoted by \(\hat{\cdot}\), to (3.4) and (3.5) this system can be solved explicitly to the result

\[
\begin{aligned}
\hat{u}^+(y) &= \int_0^\infty k_+(y, s)\hat{f}^+(s)ds, \quad y > 0, \\
\hat{u}^-(y) &= \int_{-\infty}^0 k_-(y, s)\hat{f}^-(s)ds, \quad y < 0, \\
\hat{\rho}_E(y) &= e^{-\frac{c}{\sqrt{\xi}}|y|}\hat{\rho}, \\
\hat{\rho} = \frac{1}{m} \left( \hat{h} - \int_0^\infty e^{-\omega_+ s/\sqrt{\xi}} \hat{f}^+(s)ds - \int_{-\infty}^0 e^{-\omega_- s/\sqrt{\xi}} \hat{f}^-(s)ds \right).
\end{aligned}
\]

(3.6) (3.7) (3.8) (3.9)

Here we used the abbreviations

\[
k_\pm(|y|, |s|) := \frac{1}{2\omega_\pm \sqrt{\omega_\pm}} \left( e^{-\omega_- |\xi - |s|/\sqrt{\xi}} - e^{-\omega_+ (|y| + |s|)/\sqrt{\xi}} \right),
\]

\[
\omega = \omega(\lambda, \xi, y) = \sqrt{\lambda + 1 + c(y)|\xi|^2},
\]

\[
\omega_\pm = \omega_\pm(\lambda, \xi) = \sqrt{\lambda + 1 + c_\pm |\xi|^2},
\]

\[
m = \lambda + 1 + a_+ \sqrt{\xi} \omega_+ + a_- \sqrt{\xi} \omega_-.
\]

(iii) The desired regularity for \(u\) follows by the fact that it solves the heat equation in \(\mathbb{R}^{n+1}\). The function \(1/m\) represents the principal symbol of the linearization for the classical Stefan problem. We denote by \(Op(1/m)\) the associated operator, i.e. \(Op(1/m)u = (1/m)\hat{u}\). Note also that \(\hat{f}^\pm \in L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n+1}))\) implies that the inverse Fourier-Laplace transform of \(\int_0^\infty e^{-\omega_+ s/\sqrt{\xi}} \hat{f}^+(s)ds\) and \(\int_{-\infty}^0 e^{-\omega_- s/\sqrt{\xi}} \hat{f}^-(s)ds\) belongs to the space

\[
o W_p^{1/2 - 1/2p}(\mathbb{R}_+; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; W^{1-1/p}(\mathbb{R}^n))
\]

(see [8, pages 15–16]). Thus, if we can show that \(Op(1/m)\) is an isomorphism between the right spaces, then the regularity for \(\rho\) and \(\rho_E\) is also clear.
LEMMA 3.2. Let $1 < p < \infty$. Then $\text{Op}(1/m)$ maps the space
\[ 0W^{1/2-1/2p}(\mathbb{R}^+; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}^+; W^{1-1/p}_p(\mathbb{R}^n)) \]
continuously into the space
\[ 0W^{3/2-1/2p}(\mathbb{R}^+; L_p(\mathbb{R}^n)) \cap \partial H^1_p(\mathbb{R}^+; W^{1-1/p}_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}^+; W^{2-1/p}_p(\mathbb{R}^n)). \]

The proof of this Lemma is based on an abstract result of Kalton and Weis [13, Theorem 4.4]. It essentially follows by the facts that both, the Poisson operator \((-\Delta)^{1/2}\), corresponding to the symbol $|\xi|$, and the operator $\partial_t + 1$, corresponding to $\lambda + 1$, admit a bounded $\mathcal{H}$-calculus on $\mathcal{H}_p^\infty(\mathbb{R}^+; \mathcal{K}_p^s(\mathbb{R}^n))$ with $H^\infty$-angles $\phi_{(-\Delta)^{1/2}} = 0$ and $\phi_{\partial_t + 1} = \pi/2$, respectively, where $1 < p < \infty$ and $r, s \in \mathbb{R}$. Here $\mathcal{K} \in \{H, W\}$, i.e. by $\mathcal{K}_p^s$ we mean either the space $H^s_p$ or $W^s_p$. We refer to [22] for the details.

As a consequence we obtain that $\rho$ and therefore also $\rho_E$ possess the regularity we claimed in Theorem 3.1, and the proof of Theorem 3.1 is completed.

4. The two-phase problem, local existence. Let $E_T = E_T^1 \times E_T^2$ denote the regularity class of the solution $(v, \rho)$ and let $F_T = F_T^1 \times F_T^2$ denote the class of the data $(f, h)$. By $\partial E_T$, $\partial F_T$ we mean the corresponding spaces with zero time trace. It will be convenient to split the solution in a part with zero time trace plus a remaining part taking care of the non zero traces. For this purpose we employ Theorem 3.1, which gives us a solution $(v^*, \rho^*)$ for the linear problem (3.1) with given data
\[ (f, h, v_0, \rho_0) = (0, h^*, v_0, \rho_0) \quad \text{where} \quad h^*(t) := e^{t\Delta}H(v_0, w_0). \]

It is a consequence of the assumptions on the initial data that the data in the line above satisfy the assumptions (a)–(e) of Theorem 3.1. Thus, $(v^*, \rho^*) \in E_T$ is well-defined and it suffices to study the reduced nonlinear problem
\[
\begin{cases}
(\partial_t - c\Delta)\bar{v} = F_0(\bar{v}, \bar{w}) & \text{in } J \times \mathbb{R}^{n+1}, \\
\gamma \bar{v}^\pm = 0 & \text{on } J \times \mathbb{R}^n, \\
\partial_t \bar{\rho} + [c\gamma \partial_y(\bar{v} - aw)] = H_0(\bar{v}, \bar{w}) & \text{on } J \times \mathbb{R}^n, \\
\bar{v}(0) = 0 & \text{in } \mathbb{R}^{n+1}, \\
\bar{\rho}(0) = 0 & \text{in } \mathbb{R}^n,
\end{cases}
\]
with
\[ F_0(\bar{v}, \bar{w}) := F(\bar{v} + v^*, \bar{w} + w^*), \quad H_0(\bar{v}, \bar{w}) := H(\bar{v} + v^*, \bar{w} + w^*) - h^*, \]
where $\bar{w}$ and $w^*$ are extensions of $\bar{\rho}$ and $\rho^*$, respectively, satisfying equation (3.2). Here we observe that
\[ H_0(\bar{v}, \bar{w}) \in \partial F_T^2 \]
for all functions $(\bar{v}, \bar{w}) \in \partial E_T^1 \times \partial E_T^1$ with $\|\partial_y(\bar{w} + w^*)\|_\infty \leq 1/2$. Thanks to this and Theorem 3.1(ii), the reduced nonlinear problem (4.1) can now be rephrased as a fixed point equation
\[ (\bar{v}, \bar{\rho}) = K_0(\bar{v}, \bar{\rho}) := S_T\{F_0(\bar{v}, \bar{w}), H_0(\bar{v}, \bar{w})\} \quad \text{in } \partial E_T, \quad (4.3) \]
where $S_T$ is the solution operator of the linear problem defined in (3.3).

The advantage of applying the fixed point argument in the zero trace space $0E_T$ lies in the fact that the embedding constant of the embedding

$$0E_T \hookrightarrow 0\text{BUC}(J;\text{BUC}^1(\mathbb{R}^{n+1}))$$

does not depend on the length of the time interval $J = (0, T)$. Moreover, according to Theorem 3.1(ii), the norm of the solution operator $S_T$ is independent of $T$ as well. This enables us to choose $T$ as small as we wish for without having the constants blowing up.

In order to show that $K_0$ is a contraction, mapping a small Ball of radius $r$ into itself, we have to provide suitable estimates of the nonlinearities $F$ and $H$ as defined in (2.3) and (2.3). By an inspection of the single terms appearing in the expressions of $F$ and $H$ we see that there are basically three different kind of terms:

- Terms that will become small by choosing $r$ small,
- terms that will become small by choosing $T$ small,
- terms that will become small by the assumptions on the initial data.

This allows us to apply the contraction mapping principle in order to deduce the following result.

**Theorem 4.1.** Fix $p > n + 3$. Then there is a number $\eta > 0$ such that the following holds: Given $(v_0, \rho_0) \in W_p^{2-2/p}(\mathbb{R}^{n+1}) \times W_p^{2-2/p}(\mathbb{R}^n)$ with

$$\gamma v_0^\pm = 0, \quad \pm v_0^\pm > 0 \text{ on } \mathbb{R}^{n+1}_\pm, \quad a_\pm > 0, \quad \text{and} \quad (4.4)$$

$$\|\rho_0\|_{\text{BUC}^1(\mathbb{R}^n)} + \left\| \frac{\gamma \partial_y v_0}{1 + \gamma \partial_y v_0} - a \right\|_{\text{BUC}(\mathbb{R}^n)} \leq \eta, \quad \text{where}$$

$$w_0 := e^{-|v(1-\Delta)^{1/2}} \rho_0, \quad a_\pm := \frac{\gamma \partial_y v_0^\pm(0,0)}{1 + \gamma \partial_y w_0^\pm(0,0)}, \quad (4.5)$$

there exists $T = T(v_0, \rho_0)$ and a unique solution $(v, \rho) \in E_T$ for the nonlinear problem (2.1).

5. Analyticity. To prove the analyticity of the solutions to (2.1) we employ a scaling argument and the implicit function theorem. Roughly speaking, this means we consider the translated and dilated solution

$$\tau_{\lambda,\mu} v(t, x, y) := v(\lambda t, x + t\mu, y), \quad \tau_{\lambda,\mu} \rho(t, x) := \rho(\lambda t, x + t\mu)$$

for $(\lambda, \mu) \in (1-\delta, 1+\delta) \times \mathbb{R}^n$ with $\delta > 0$ sufficiently small, and show by an application of the implicit function theorem that the dependence of $\tau_{\lambda,\mu} v$ and $\tau_{\lambda,\mu} \rho$ on $\lambda$ and $\mu$ is analytic. In order to apply this method to a quasilinear system such as the Stefan problem requires the following three ingredients:

(i) Maximal regularity for the linearization.

(ii) The nonlinearities are real analytic maps, that is in our situation

$$(F, H) \in C^\infty(G_T, F_T)$$

for an appropriate open subset $G_T \subseteq E_T^1 \times E_T^1$.

(iii) The nonlinearities commute with translations in space and dilations in time, i.e.

$$\tau_{\lambda,\mu} F = F \tau_{\lambda,\mu}, \quad \tau_{\lambda,\mu} H = H \tau_{\lambda,\mu}.$$
In our situation (i) is an immediate consequence of Theorem 3.1. Condition (ii) follows essentially from an inspection of the representations (2.3) and (2.3) for $F$ and $H$, respectively. Indeed, it can be shown that (ii) is satisfied for

$$G_T = \{(v, w) \in E_T^1 \times E_T^1 : \|\partial_y w\|_{\infty} \leq 1/2\}.$$  

On the other hand (iii) can be easily seen by these representations, since there does not appear a time derivative.

Here we also employ the splitting

$$(v, \rho) = (\bar{v}, \bar{\rho}) + (v^*, \rho^*)$$

with $(\bar{v}, \bar{\rho}) \in \partial E_T^1 \times_0 E_T^2$, and $(v^*, \rho^*)$ taking care of the non zero traces, as introduced in the previous section. We focus on the first summand and suppose that the analyticity of $(v^*, \rho^*)$ is already proved, which, for instance, can be obtained as well by an application of the implicit function theorem.

Next, let $\Lambda \subseteq (1 - \delta, 1 + \delta) \times \mathbb{R}^n$ and $\partial B_T^1(0, r) \times \partial B_T^2(0, r) \subseteq \partial E_T^1 \times_0 E_T^2$, where $\partial B_T^1(0, r)$ denotes the ball with center 0 and radius $r$. Further, we introduce the nonlinear map

$$\Psi_0 : \partial B_T^1(0, r) \times \partial B_T^2(0, r) \times \Lambda \rightarrow \partial F_T$$

$$\Psi_0((u, \sigma), (\lambda, \mu)) := \begin{pmatrix} (\partial_t - \lambda c\Delta)u - F_{\lambda, \mu}(u, \sigma) \\ \partial_t \rho + \lambda |c\gamma \partial_y (u - T(\lambda, \mu)\sigma)| - H_{\lambda, \mu}(u, \sigma) \end{pmatrix},$$

where

$$F_{\lambda, \mu}(u, \sigma) := \lambda F(u + \tau_{\lambda, \mu} v^*, T(\lambda, \mu)\sigma + \tau_{\lambda, \mu} w^*) + (|\nabla u|, \mu)$$

$$H_{\lambda, \mu}(u, \sigma) := \lambda H(u + \tau_{\lambda, \mu} v^*, T(\lambda, \mu)\sigma + \tau_{\lambda, \mu} w^*) - \tau_{\lambda, \mu} h^* + (|\nabla \sigma|, \mu)$$

and $T(\lambda, \mu) := \tau_{\lambda, \mu} (\tau_{1/\lambda, -\mu} \sigma)_E$. The analyticity of $F$ and $H$ implies that also

$$\Psi_0 \in C^\omega(\partial B_T^1(0, r) \times \partial B_T^2(0, r) \times \Lambda, \partial F_T).$$

It readily follows that, if $(\bar{v}, \bar{\rho})$ solves (2.1) then

$$(u, \sigma) = (\tau_{\lambda, \mu} \bar{v}, \tau_{\lambda, \mu} \bar{\rho})$$

satisfies $\Psi_0((u, \sigma), (\lambda, \mu)) = 0$. Therefore, by utilizing the results of the last section it can be shown that for $r, \delta, T > 0$ small enough $\Psi_0$ is well defined. It turns out that the Fréchet derivative of $\Psi_0$ with respect to $(\bar{v}, \bar{\rho})$ at $(\lambda, \mu) = (1, 0)$ is given by

$$D_1 \Psi_0((\bar{v}, \bar{\rho}), (1, 0))\bar{u} = U[\bar{u}, \bar{\sigma}] - (DF_0(\bar{v}, \bar{\rho}_E), DH_0(\bar{v}, \bar{\rho}_E))[\bar{u}, \bar{\sigma}_E]$$

for $(\bar{u}, \bar{\sigma}) \in \partial E_T$, where

$$U[\bar{u}, \bar{\sigma}] := ((\partial_t - c\Delta)\bar{u}, \partial_t \bar{\sigma} + |c\gamma \partial_y (\bar{u} - a\bar{\sigma}_E)|)$$

and $F_0$ and $H_0$ are defined in (4.2). The proof of the existence also shows that the respective norms of the Fréchet derivatives of $F_0$ and $H_0$ are small for $(\bar{v}, \bar{\rho}_E) \in \partial B_T^1(0, r) \times \partial E_T^2$. 

Equadiff-11. Analytic solutions for the Stefan problem
\(aB^1_T(0, r)\) if we suppose that \(r, T\) are sufficiently small. This fact and since \(U\) represents exactly the linearization given in (3.1), Theorem 3.1 implies
\[
D_1 \Psi_0((\bar{v}, \bar{\sigma}), (1, 0)) \in \text{Isom}(aE_T, aF_T).
\]
Thus the analyticity of \(\Psi_0\) and the implicit function theorem imply the existence of an open neighborhood of \((1, 0)\) in \((1 - \delta, 1 + \delta) \times \mathbb{R}^n\), again denoted by \(\Lambda\), such that
\[
[(\lambda, \mu) \mapsto (\tau_{\lambda, \mu} \bar{v}, \tau_{\lambda, \mu} \bar{\sigma})] \in C^\omega(\Lambda, aE_T).
\]
The analyticity of the solution \((u, \Gamma)\) of the classical Stefan problem (1.1) is now essentially a consequence of (5.1). This completes the proof of Theorem 1.1.

Acknowledgments. The work of the second author was mainly developed during his stay at the Department of Mathematics at Vanderbilt University. He would like to express his gratitude to the Department, and especially to Gieri Simonett for giving him the opportunity of this stay, for the hospitality, and for always fruitful discussions during his stay.

REFERENCES


