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SEMILINEAR PROBLEMS PERTURBED THROUGH THE BOUNDARY CONDITION*

JORGE GARCÍA-MELIÁN[†], JULIO D. ROSSI[‡], AND JOSÉ SABINA DE LIS[§]

Abstract. We analyze the semilinear diffusion equation $\Delta u = a(x)u^p$ in a smooth bounded domain Ω subjected to the boundary condition $\partial u/\partial \nu = \lambda u$, where ν is the outward unit normal to $\partial\Omega$, λ is a real parameter. The coefficient $a(x)$ is a nonnegative weight function, which could even vanish in a whole smooth subdomain Ω_0 of Ω . We consider both cases $p > 1$ and $0 < p < 1$, and give a detailed description of the existence, uniqueness or multiplicity and asymptotic behavior of nonnegative solutions. As an additional special feature, the adherence of the portions $\Omega \cap \partial\Omega_0$, $\partial\Omega \cap \partial\Omega_0$ of the boundary of Ω_0 are allowed to meet each other in a smooth manifold.

Key words. Bifurcation, Steklov problem, boundary blow-up, perturbation of domains.

AMS subject classifications. 35J25, 35B40.

1. Introduction. In this note we are giving a summary of the issues of existence and uniqueness – alternatively multiplicity – of positive solutions to the semilinear problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a $C^{2,\alpha}$ bounded domain, $a \in C^\alpha(\bar{\Omega})$, $a(x) \geq 0$.

The main features of problem (1.1) are: a) the presence of the bifurcation parameter λ in the boundary condition, b) the possibility that $a \geq 0$ can vanish in a whole subdomain Ω_0 of Ω and c) the two regimes of the problem: “regular” and “degenerate”, corresponding to the ranges $p > 1$ and $0 < p < 1$ of the exponent p respectively. In particular, we are paying special attention to the study of the asymptotic profiles of the solutions when the parameter λ tends to critical values λ_c where bifurcation phenomena occur. In the degenerate case, the study of the regions where the solutions u vanish identically (“dead cores”) will be an additional issue to deal with.

Problem (1.1) constitutes another example of the group of models where a “dissipative” (“absorption”) mechanism is in competition with another one of “production” (“radiation”). The logistic problem under either of the three classical boundary conditions is the paradigmatic example of this setting (see [8], [2], [3], [1] for the discussion of phenomena which are related to the ones covered here).

In the rest of the note we will assume that the coefficient $a(x)$ verifies the following hypothesis:

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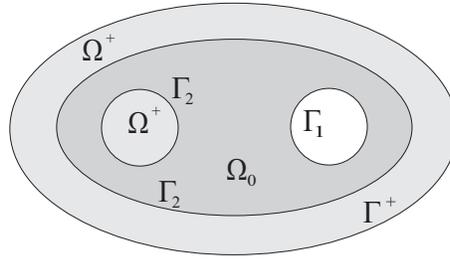


FIG. 1.1. A possible configuration for an Ω_0 satisfying $H)_s$.

H) $a \in C^\alpha(\bar{\Omega})$ is nonnegative and nontrivial. Moreover, it is either strictly positive in Ω or $a \equiv 0$ in a subdomain $\Omega_0 \subset \Omega$ of class $C^{2,\alpha}$.

For immediate reference we are fixing the following notations: $\Gamma_2 = \Omega \cap \partial\Omega_0$, $\Gamma_1 = \partial\Omega_0 \setminus \Gamma_2$. In addition, $\Omega^+ = \{x \in \Omega : a(x) > 0\}$, $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$. Notice that since $a(x)$ is nontrivial then both Γ_2 and Ω^+ must always be non void meanwhile $\Omega \cap \partial\Omega^+ = \Gamma_2$.

In our previous works [4], [5], an important part of the results was obtained under the simplifying assumption,

$$H)_s \quad \bar{\Gamma}_2 = \Gamma_2,$$

the bar standing for “adherence”. This means that Γ_2 is a closed $N - 1$ dimensional manifold and, more importantly, that Γ_1 is separated away from Γ_2 ($\text{dist}(\Gamma_1, \Gamma_2) > 0$). In particular, Ω^+ also defines a $C^{2,\alpha}$ subdomain of Ω , while $\Omega_0, \Omega^+, \Gamma_1, \Gamma_2, \Gamma^+$ consist only of a finite number of connected pieces (FIG.1.1). We are reviewing the above mentioned results in Sections 3 and 4.

In the present work some of the main statements in [4] are significantly extended to cover more natural configurations for Γ_1, Γ_2 in $\partial\Omega_0$ than in $H)_s$. In particular, we allow $\bar{\Gamma}_2 \cap \Gamma_1 \neq \emptyset$ (see details in Sections 3, 4 and Figure 3.2).

REMARK 1. The connectedness condition in Ω_0 can be relaxed without significant changes in the results.

2. Eigenvalue problems. In the analysis of (1.1) several eigenvalue problems which are non-standard appear. In fact, some of them involve the eigenvalue in the boundary condition while other ones exhibit a non usual regime for the coefficients.

A first eigenvalue problem is

$$\begin{cases} \Delta\phi = \lambda\phi & x \in \Omega_0 \\ \frac{\partial\phi}{\partial\nu} = \mu\phi & x \in \partial\Omega, \end{cases} \quad (2.1)$$

$\mu > 0$ regarded as a parameter. The existence of a principal eigenvalue $\lambda_1(\mu)$ to (2.1) and its qualitative behavior with respect to μ was first studied in [7] and extended in several aspects in [4]. The corresponding principal eigenfunctions ϕ are a source of sub and supersolutions for problem (1.1) in the regime $p > 1$.

The description of the set of positive solutions for problem (1.1) when $\Omega_0 \neq \emptyset$ requires the introduction of several critical values for λ , characterized by means of some eigenvalue problems. Specifically, we define $\lambda = \sigma_1$ as the principal eigenvalue of the mixed Steklov-

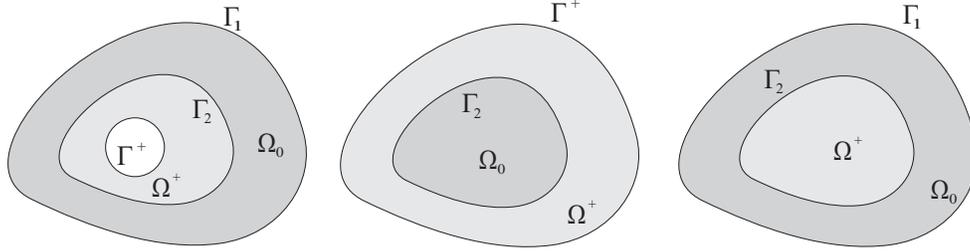


FIG. 2.1. Several configurations for Ω_0 and Ω^+ : $\Gamma_1, \Gamma^+ \neq \emptyset, \Gamma_1 = \emptyset, \Gamma^+ \neq \emptyset$ and $\Gamma_1 \neq \emptyset, \Gamma^+ = \emptyset$.

type eigenvalue problem:

$$\begin{cases} \Delta\phi = 0 & x \in \Omega_0 \\ \phi = 0 & x \in \Gamma_2 \\ \frac{\partial\phi}{\partial\nu} = \lambda\phi & x \in \Gamma_1. \end{cases} \tag{2.2}$$

Since it could happen that $\Gamma_1 = \emptyset$, we set in that case $\sigma_1 := \infty$ (FIG. 2.1).

In the same way we introduce $\lambda = \sigma_1^+$, which is the corresponding principal eigenvalue of the problem:

$$\begin{cases} \Delta\phi = 0 & x \in \Omega^+ \\ \phi = 0 & x \in \Gamma_2 \\ \frac{\partial\phi}{\partial\nu} = \lambda\phi & x \in \Gamma^+. \end{cases} \tag{2.3}$$

We also set $\sigma_1^+ := \infty$ whenever $\Gamma^+ = \emptyset$.

We refer to [4] for a detailed study of the properties of the eigenvalue problems (2.2), (2.3) and some other related problems. We remark that σ_1^+ has to be suitably defined in case Ω^+ is not connected. In the first of the configurations in FIG. 2.1, both σ_1 and σ_1^+ are finite. One of them becomes infinite in the rest of the configurations.

3. The regime $p > 1$. We are next describing the main features of problem (1.1) for the regular case where $p > 1$. In THEOREMS 3.1 to 3.3 it is assumed that $a(x)$ satisfies H) together with the “separation” hypothesis $H)_s$. In the first statement the value $\sigma_1 = \infty$ (see Section 2) is also used when referring to the case $a(x) > 0$ in Ω . Such results are collected in [4]. The section concludes with the study of the case where Γ_1 and $\bar{\Gamma}_2$ meet in a nontrivial way.

THEOREM 3.1. *Assume $a(x)$ satisfies H), $H)_s$ while $p > 1$. Then problem (1.1) admits positive solutions $u \in C^{2,\alpha}(\bar{\Omega})$ if and only if:*

$$0 < \lambda < \sigma_1 \leq \infty,$$

the solution $u = u_\lambda(x)$ being unique in that case. Moreover, the mapping $\lambda \mapsto u_\lambda$ is increasing and real analytic when considered with values in $C^{2,\alpha}(\bar{\Omega})$, and u_λ is globally attractive (among the positive solutions). Finally, $u_\lambda \rightarrow 0$ as $\lambda \rightarrow 0+$, while $|u_\lambda|_{\infty,\Omega} \rightarrow \infty$ when $\lambda \rightarrow \sigma_1$.

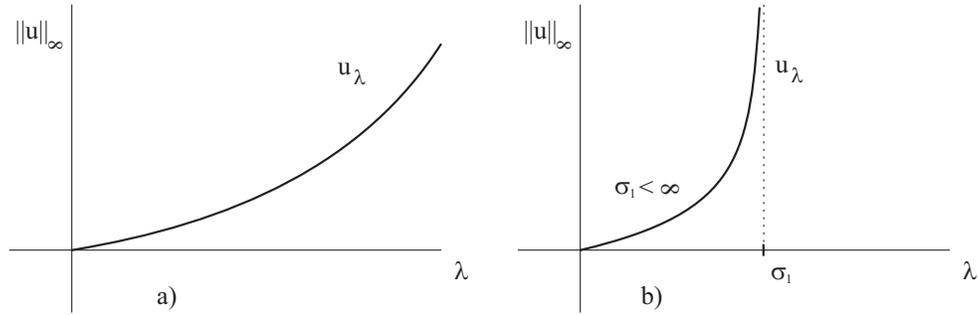


FIG. 3.1. a) $a > 0$ in Ω or $\Omega_0 \neq \emptyset$ but $\sigma_1 = +\infty$, and b) $\sigma_1 < +\infty$.

REMARK 2. THEOREM 3.1 allows the possibility that $a(x)$ vanishes somewhere on $\partial\Omega$ in the case where $a(x)$ is positive in Ω ($\Omega_0 = \emptyset$), also permitting the existence of zeros for $a(x)$ in Γ^+ if both Ω_0 and Γ^+ are nonempty ($a(x) \equiv 0$ on $\Gamma_1!$).

It has been established that if $\Omega_0 \neq \emptyset$ and $\sigma_1 < \infty$, positive solutions disappear when λ crosses the value σ_1 . The next results explain this interruption by analyzing the singularities developed by the solutions. Even if $\sigma_1 = \infty$, the solution u_λ approaches a finite profile in Ω when λ tends to infinity.

THEOREM 3.2. Assume $a(x) > 0$ for every $x \in \Omega$, or $\Omega_0 \neq \emptyset$ but $\sigma_1 = \infty$. Then the solution u_λ of (1.1) satisfies $u_\lambda \rightarrow u$ in $C^{2,\alpha}(\Omega)$ as $\lambda \rightarrow \infty$ where $u(x)$ is the minimal solution to the singular boundary value problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ u = \infty & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Moreover, we have the following estimate of the asymptotic growth rate:

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-\frac{2}{p-1}} \sup u_\lambda \geq |a|_\infty^{-\frac{1}{p-1}}.$$

THEOREM 3.3. Under the condition $\sigma_1 < \infty$ and hence $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega \neq \emptyset$, the solution u_λ to (1.1) satisfies $u_\lambda \rightarrow \infty$ uniformly in $\bar{\Omega}_0$ as $\lambda \rightarrow \sigma_1^-$. In addition, $u_\lambda \rightarrow u$ in $C^{2,\alpha}(\Omega^+)$ where $u(x)$ is the minimal solution to the singular boundary value problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\ u = \infty & x \in \Gamma_2 \\ \frac{\partial u}{\partial \nu} = \sigma_1 u & x \in \Gamma^+, \end{cases} \tag{3.2}$$

provided that $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega \neq \emptyset$, or $u(x)$ is the minimal solution to the problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\ u = \infty & x \in \partial\Omega^+, \end{cases} \tag{3.3}$$

in case $\Gamma^+ = \emptyset$.

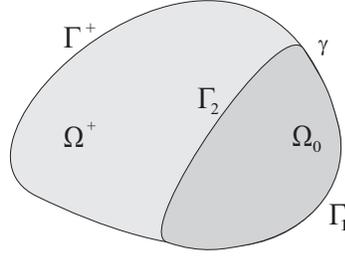


FIG. 3.2. Configuration for Ω_0 with $\bar{\Gamma}_2$ and Γ_1 contacting in a smooth manifold γ .

REMARK 3. Observe that we prove the existence of nontrivial solutions of (3.2). This is a novelty in view of the boundary condition on Γ^+ not previously treated in the literature (see specially the corresponding problem in THEOREM 3.4). On the other hand, suitable conditions on the weight $a(x)$ can be given in order to obtain uniqueness of positive solutions to the singular problems (3.1), (3.2) and (3.3).

We are next dealing with the hypothesis H) under a less restrictive assumption than $H)_s$ regarding the separation between $\Gamma_1, \bar{\Gamma}_2$. Namely, it will be assumed that $\Omega_0 \subset \Omega$ is a C^3 subdomain such that (recall that $\Gamma_2 = \Omega \cap \partial\Omega_0, \Gamma_1 = \partial\Omega \cap \partial\Omega_0$),

$H)_m$ $\Gamma_1, \bar{\Gamma}_2$ are nonempty $N - 1$ -dimensional manifolds having as common boundary γ , a $(N - 2)$ -dimensional closed submanifold of $\partial\Omega$ (FIG. 3.2).

The existence, under these conditions on Ω_0 , of a unique principal weak eigenvalue $0 < \sigma_1 < \infty$ to (2.2) with a positive associated eigenfunction $\phi_1 \in H^1(\Omega_0) \cap W^{2,s}(\Omega_0)$, $1 < s < 4/3$, can be ensured by variational methods (see [5]).

Our main statement concerning this new framework for problem (1.1) essentially asserts that THEOREMS 3.1 and 3.3 are still valid even when Γ_1 and $\bar{\Gamma}_2$ meet each other.

THEOREM 3.4. Suppose that $a \in C^\alpha(\bar{\Omega})$ satisfies H) together with condition $H)_m$. Then problem (1.1) admits a weak positive solution in $H^1(\Omega)$ if and only if,

$$0 < \lambda < \sigma_1.$$

Such solution is unique and defines indeed a classical solution in $C^{2,\alpha}(\bar{\Omega})$. As a mapping of $\lambda \in (0, \sigma_1)$, u_λ is increasing, smooth and bifurcates from zero at $\lambda = 0$. Moreover,

$$u_\lambda \rightarrow u,$$

as $\lambda \rightarrow \sigma_1^-$ in $C^{2,\alpha}(\Omega^+) \cap C^{1,\alpha}(\Omega^+ \cup T)$ for every compact $T \subset \Gamma^+$ while,

$$u_\lambda \rightarrow \infty,$$

as $\lambda \rightarrow \sigma_1^-$ uniformly on each compact set $K \subset \Gamma_2$. In particular:

$$\lim_{\text{dist}(x, K) \rightarrow 0} u(x) = \infty,$$

and thus u defines a classical solution to problem (3.2),

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\ u = \infty & x \in \Gamma_2 \\ \frac{\partial u}{\partial \nu} = \sigma_1 u & x \in \Gamma^+. \end{cases}$$

REMARK 4. In contrast to the more regular case of the hypothesis $H)_s$, the approach to achieve existence in THEOREM 3.4 is variational. In this regard, existence can be also obtained if Ω_0 is merely Lipschitz (cf. [4, Theorem 2]). An additional regularity argument then shows that any weak $H^1(\Omega)$ solution to (1.1) lies indeed in $L^\infty(\Omega)$ and hence its smoothness improves up to be a classical solution (see further details in [6]). On the other hand, it is a more subtle question to decide which is the possible asymptotic profile of u_λ in the interphase γ as $\lambda \rightarrow \sigma_1^-$.

4. Degenerate regime. Let us now analyze problem (1.1) in the degenerate range $0 < p < 1$. The results we describe in what follows are contained in the work [5] with the exception of those involving the condition $H)_m$. We are denoting $\alpha_1 = \min\{p, \alpha\}$.

THEOREM 4.1. *Assume that the coefficient $a \in C^\alpha(\overline{\Omega})$ satisfies $a(x) > 0$ for every $x \in \Omega$. Then:*

- i) [Existence] *Problem (1.1) admits at least one nonnegative solution $u \in C^{2,\alpha_1}(\overline{\Omega})$, $u \neq 0$, for every $\lambda > 0$ and no nonnegative solutions exist if $\lambda \leq 0$.*
- ii) [Bifurcation from infinity] *For certain $\lambda_0 > 0$ small enough and $0 < \lambda < \lambda_0$ there exists a unique positive solution $u_\lambda \in C^{2,\alpha}(\overline{\Omega})$. The mapping $\lambda \mapsto u_\lambda$ is real analytic in $(0, \lambda_0)$ with values in $C^{2,\alpha}(\overline{\Omega})$. Moreover, it is decreasing and satisfies:*

$$\lim_{\lambda \rightarrow 0^+} \lambda^{\frac{1}{1-p}} u_\lambda(x) = \left(\frac{1}{|\partial\Omega|} \int_\Omega a \right)^{\frac{1}{1-p}}$$

where the limit is taken in $C^{2,\alpha}(\overline{\Omega})$.

- iii) [L^∞ estimate] *There exist constants $\lambda_1 > 0$, $C > 0$ such that every nonnegative solution u corresponding to $\lambda \geq \lambda_1$ satisfies:*

$$0 \leq u(x) \leq C\lambda^{-\frac{2}{1-p}}.$$

- iv) [Dead core formation] *Every nonnegative solution $u_\lambda \neq 0$ corresponding to $\lambda \geq \lambda_2$, for a certain $\lambda_2 > 0$, develops a dead core $\mathcal{O}_\lambda = \{u_\lambda(x) = 0\}$ such that $\mathcal{O}_\lambda \rightarrow \Omega$ as $\lambda \rightarrow \infty$. More precisely:*

$$\{x : \text{dist}(x, \partial\Omega) \geq d(\lambda)\} \subset \mathcal{O}_\lambda,$$

where $d(\lambda) \rightarrow 0^+$ as $\lambda \rightarrow \infty$. Furthermore, $d(\lambda)$ can be chosen as $d(\lambda) = \frac{K}{\lambda}$, for a certain constant $K > 0$, provided that $a(x) > 0$ on $\partial\Omega$.

An important feature of the degenerate regime is that it exhibits multiple solutions when λ is sufficiently large. This is shown in the next result.

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{2,\alpha}$ domain whose boundary $\partial\Omega$ splits in k connected components, while $a \in C^\alpha(\overline{\Omega})$ is positive in Ω . Then problem (1.1) admits at least $2^k - 1$ nonnegative nontrivial solutions when λ is large enough.*

In view of this result, a question naturally arises: is the non connectedness of $\partial\Omega$ responsible for the multiplicity of solutions? The answer – in the negative – can be found in the next result, where Ω is a ball, hence $\partial\Omega$ is connected.

THEOREM 4.3. *Consider problem (1.1) in a ball B of \mathbb{R}^N , with the coefficient $a(x)$ radial and positive. Then problem (1.1) admits for every $\lambda > 0$ a radial nonnegative solution $u \neq 0$. Such solution is unique for $0 < \lambda < \lambda_0$. Moreover:*

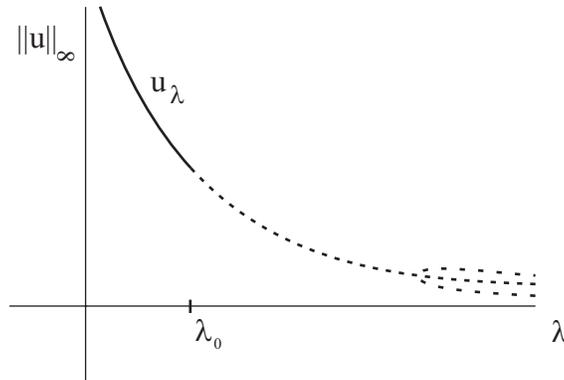


FIG. 4.1. Bifurcation diagram for $a(x) > 0$: the continuous line means uniqueness.

i) There exists a unique radial nonnegative solution $u_\lambda \neq 0$ for large λ that satisfies:

$$\text{dist}(\mathcal{O}_\lambda, \partial B) \sim \beta\lambda^{-1}, \quad u_\lambda(1) \sim A\beta^\beta\lambda^{-\beta},$$

as $\lambda \rightarrow +\infty$ where $\beta = 2/(1-p)$, $A = [\beta(\beta-1)]^{-1/(1-p)}$ and $\mathcal{O}_\lambda = \{x \in B : u_\lambda = 0\}$.

ii) There exists $\lambda_3 > 0$ such that problem (1.1) admits a solution $u \neq 0$, nonnegative and nonradial for every $\lambda \geq \lambda_3$.

REMARK 5. Suppose that $a(x)$ satisfies H) with a nonempty domain Ω_0 so that $\sigma_1 = \infty$. Since this is equivalent to $\bar{\Omega}_0 \subset \Omega$ ($\Gamma_1 = \emptyset$, see Section 2) then $H)_s$ is satisfied and it is possible to deduce for problem (1.1) the same conclusions obtained in THEOREM 4.1. Namely:

- i) For $\lambda > 0$, problem (1.1) admits at least a nonnegative nontrivial solution $u \in C^{2,\alpha_1}(\bar{\Omega})$, and no solutions exist if $\lambda \leq 0$.
- ii) There exists a unique positive solution $u \in C^{2,\alpha}(\bar{\Omega})$ for $0 < \lambda < \lambda_0$ which bifurcates from infinity in $\lambda = 0$, with $u_\lambda(x) \sim \left(\frac{1}{|\partial\Omega|} \int_\Omega a\right)^{\frac{1}{1-p}} \lambda^{-\frac{1}{1-p}}$ as $\lambda \rightarrow 0+$.
- iii) For large λ , nonnegative solutions u verify the estimate $0 \leq u(x) \leq C\lambda^{-\frac{2}{1-p}}$, where the constant C does not depend on u .
- iv) Also for large λ , all nonnegative solutions u develop a dead core \mathcal{O}_λ which satisfies $\{x : \text{dist}(x, \partial\Omega) \geq d(\lambda)\} \subset \mathcal{O}_\lambda = \{u_\lambda(x) = 0\}$, $d(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

However, the situation is completely different to the regular regime $p > 1$ when $\sigma_1 < \infty$ ($\Gamma_1 \neq \emptyset$). In strong contrast with the regular regime $p > 1$, the eigenvalue σ_1^+ plays an important rôle in the degenerate case $0 < p < 1$. The next statement gives precise information on problem (1.1) when $\sigma_1 < \infty$.

THEOREM 4.4. Suppose that $a(x)$ satisfies H) with $\sigma_1 < \infty$ and assume either of the conditions $H)_s$ or $H)_m$. Then there exists at least one nonnegative nontrivial solution $u \in C^{2,\alpha_1}(\bar{\Omega})$, $\alpha_1 = \min\{p, \alpha\}$, with the same properties as the corresponding nonnegative solutions described in THEOREM 4.1 whenever $0 < \lambda < \sigma_1$. Moreover:

- i) All nonnegative solutions u corresponding to $\lambda \geq \sigma_1$ satisfy $u \equiv 0$ in Ω_0 .
- ii) If $\sigma_1^+ = \infty$, there do not exist nonnegative nontrivial solutions for $\lambda \geq \sigma_1$.
- iii) If $\sigma_1^+ < \infty$, nonnegative nontrivial solutions can only occur in the range $\lambda > \sigma_1^+$.

In particular, such solutions cannot exist for λ verifying:

$$\sigma_1 \leq \lambda \leq \sigma_1^+,$$

assuming $\sigma_1 \leq \sigma_1^+$ (see FIG. 4.2).

- iv) Assume that condition $H)_s$ holds. Then, for $\sigma_1^+ < \infty$ certain $\lambda_1 > \sigma_1^+$ exists such that problem (1.1) admits at least one nonnegative nontrivial solution u for every $\lambda \geq \lambda_1$. Such solutions satisfy the estimate:

$$0 \leq u(x) \leq C \lambda^{-\frac{2}{1-p}},$$

and develop a dead core which verifies $\mathcal{O}_\lambda \rightarrow \Omega$ as $\lambda \rightarrow \infty$ in the form described in REMARK 5-iv).

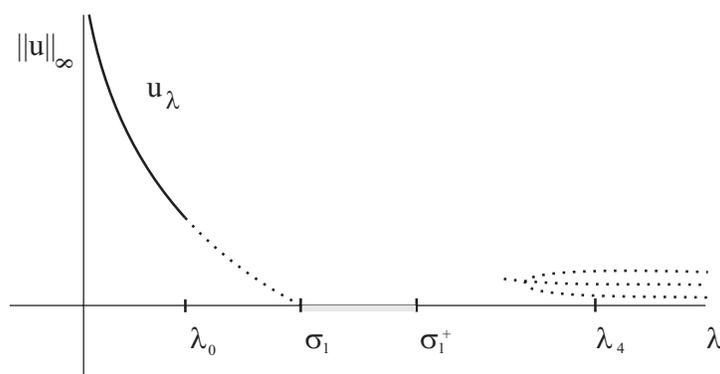


FIG. 4.2. Bifurcation when $\sigma_1 < \infty$: the continuous line stands for uniqueness.

To conclude with the description of the qualitative properties of nonnegative solutions to problem (1.1) in the degenerate case, we give sufficient conditions providing the existence of a bifurcation from $u = 0$ in $\lambda = \sigma_1$.

THEOREM 4.5. Assume that $a(x)$ satisfies $H)$ and either of the hypotheses $H)_s$ or $H)_m$. If $\sigma_1 < \infty$ and one of the following conditions holds: either $\sigma_1 \leq \sigma_1^+$ or $\sigma_1^+ < \sigma_1$ but $u = 0$ is the only nonnegative solution for $\lambda = \sigma_1$, then:

$$u_\lambda \rightarrow 0 \quad \text{as } \lambda \rightarrow \sigma_1-,$$

in $C^{2,\alpha_1}(\bar{\Omega})$. On the other hand, every possible solution develops a dead core $\mathcal{O}_\lambda \subset \Omega^+$ such that $\mathcal{O}_\lambda \rightarrow \Omega^+$ as $\lambda \rightarrow \sigma_1-$.

REMARK 6. Last assertion in THEOREM 4.5 implies that solutions u_λ of (1.1) converging to zero as $\lambda \rightarrow \sigma_1-$ develop their dead cores \mathcal{O}_λ into Ω^+ , being strictly positive in Ω_0 . This should be contrasted with the behavior of nonnegative solutions to the problem (if any) in the range $\lambda \geq \sigma_1$. In fact, the latter must be identically zero in Ω_0 (THEOREM 4.4, i)).

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