SOME BASE AXIOMS FOR TOPOLOGY INVOLVING ENUMERABILITY

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Consider the following axioms involving enumerability for a topological space $(X, \mathscr{T})$.

E_0: Every point of X is a G_δ.

E_1: For $x \in X$, $[x] = \bigcap_{n=1}^{\infty} C_n^{(x)}$ where $C_n^{(x)}$ is a closed neighborhood of x.

E_2: $(X, \mathscr{T})$ satisfies the first axiom of countability.

E_3: $(X, \mathscr{T})$ has a point countable base.

E_4: $(X, \mathscr{T})$ has a σ-point finite base.

E_5: $(X, \mathscr{T})$ has a σ-disjoint base.

E_5': $(X, \mathscr{T})$ has a uniform (point regular) base. (See definition 2.)

E_6: $(X, \mathscr{T})$ has a σ-discrete base.

E_7: $(X, \mathscr{T})$ has a countable base.

E_8: $(X, \mathscr{T})$ has a countable base $\mathscr{V}$ and the neighborhood system of each closed set has a base which is a subfamily of $\mathscr{V}$.

For $n > 2$, $E_{n+1} \to E_n$ (E_5' may be substituted for E_5 in this relation). Clearly $E_1 \to E_0$.

We will be concerned with relations of these axioms to metrization, sufficient conditions for a space satisfying E_i to satisfy E_j and examples to show the distinctness of these axioms where possible and finally the relation of some of these axioms to Moore spaces. Though there are some minor new results, the article is primarily a survey article presented with the hope of stimulating research in base axioms, particularly in regard to counter examples. For a deeper and more extensive coverage it is suggested that the reader start with [2], [3], [15], and [18].

E_0, E_1 and E_2 Spaces

E_0 and E_1 are not base axioms but their relations with E_2 are among the earlier developments of point set topology. Specifically Aleksandrov and Urysohn [4] showed that locally compact T_2, E_0 spaces are E_2 and locally countably compact T_3, E_0 spaces are E_2. Note T_3, E_0 spaces are E_1.

The author [8] has given an example of a T_2, E_0 space that is not E_1. (One point fails to be the intersection of a countable number of closed neighborhoods.) It would be interesting if there existed a homogeneous T_2, E_0 space that is not E_1[1]).

Arens [19, 77] gives an example of a denumerable T_5, E_1 space that is not E_2.

Properties of E_2 spaces are well known.

For an introduction to more recent developments see Heath [15]. For basic properties of E_0 and E_1 spaces see Aull [8].

E_3 Spaces

Recently there have been interesting developments in E_3 spaces in the Soviet Union and the United States. Some of the developments are as follows.

Theorem 1. (Aleksandrov [2]) *Locally separable T_3, E_3 spaces are metrizable.*

Theorem 2. (Ponomarev [22]) *A T_0 space is an E_3 space iff it is the continuous open image of a metric space under an S-mapping. (An S-mapping is a mapping such that the inverse images of points considered as subspaces satisfy E_7.)*

Theorem 3. (Corson and Michael 1964 [13]) *A countably compact T_2, E_3 space satisfies E_7.*

Corollary 3. (Miščenko 1961 [20]) *A compact T_2, E_3 space satisfies E_7.*

Theorem 4. (Heath [16]) *A T_3, E_3, M_1 space is metrizable. (An M_1 space is a topological space with a σ-closure preserving base. See also Ceder* [12]).

To the author's knowledge there are no non-trivial conditions for an E_2 space to be E_3 unless one would want to consider denumerability of the space non-trivial. In fact there is an example of Aleksandrov and Urysohn [4, 77] of a compact E_2, T_2, hereditary separable, hereditary Lindelöf space satisfying the countable chain condition which is not metrizable. By Corollary 3, E_3 is not satisfied.

E_4 and E_5 Spaces

Theorem 5. (Arhangel'skii [7]) *A T_3, E_4 space is metrizable iff it is perfectly normal and collectionwise normal.*

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1) Editor's note: See S. P. Franklin, A homogeneous Hausdorff E_0-space which isn't E_1, this volume, pages 125–126.

Miščenko [20] gave an example of a hereditary Lindelöf T_2, E_3 space not satisfying T_3 or E_7. It is a consequence of Theorem 5 that T_3, E_4 hereditary Lindelöf spaces satisfy E_6. However, we need a new theorem to establish that Miščenko's example is not E_4 since his example is not T_3.

Theorem 6. *A hereditary Lindelöf E_4 space satisfies E_7.*

Proof. Let the σ-point finite base be designated by $\mathscr{V} = \bigcup \mathscr{V}_n$ where each $\mathscr{V}_n$ is point finite. For fixed n there is a countable subcover $\mathscr{W}_n^1 \subset \mathscr{V}_n$ of $\bigcup\{V: V \in \mathscr{V}_n\}$. In general there is a countable subcover $\mathscr{W}_n^k$ of $\bigcup\{V: V \in \mathscr{V}_n \sim \bigcup_{i=1}^{k-1} \mathscr{W}_n^i\}$. By the point finiteness of $\mathscr{V}_n$, $\mathscr{V}_n = \bigcup_{k=1}^{\infty} \mathscr{W}_n^k$ and E_7 is satisfied.

In view of Theorem 5 it would be desirable to know if there exists a perfectly normal, collectionwise normal, E_3, T_2 space that is not metrizable and hence not E_4.

For E_5 spaces we have the following simple metrization theorem.

Theorem 7. *Every perfectly normal E_5 space satisfies E_6. Perfectly T_4, E_5 spaces are metrizable.*

Proof. Let $\mathscr{V}$ be a disjoint family of open sets. $W = \bigcup\{V: V \subset \mathscr{V}\}$ is an F_σ set, i.e. $W = \bigcup F_n$ where each F_n is closed. There exists open G_n such that $F_n \subset G_n \subset \bar{G}_n \subset W$. $\{V \cap G_n\}$ is a discrete family for each n.

In view of Theorem 7 and the complicated proof of Theorem 5 it would be highly unlikely that $E_4 = E_5$. See the end of the next section for a T_3 not T_4, E_4 space that is not E_5. Corson and Michael [13], have an example of a non-metrizable T_2, hereditarily paracompact, Lindelöf E_5 space.

E_5' Spaces and Moore Spaces

In order to proceed further we will need to review some definitions.

Definition 1. *A Moore space is a T_3 developable space. A topological space is developable if there exists a base $\mathscr{V} = \bigcup \mathscr{V}_n$ for the topology $\mathscr{T}$ such that each $\mathscr{V}_n$ covers X and for $x \in X$, $T \in \mathscr{T}$, there exists n, $n(T)$ such that if $x \in V \in \mathscr{V}_n$, then $V \subset T$. The family $\mathscr{V}$ is referred to as a development.*

Definition 2. (E_5') *A base $\mathscr{V}$ for a topological space $(X, \mathscr{T})$ is point regular or uniform if every infinite subfamily of $\mathscr{V}$, each member of this subfamily containing a given (arbitrary) point is a base at this point.*

Aleksandrov [1] showed that E_5' spaces are point countable and the base is a development such that each cover can be taken as point finite. From this Heath [17] concludes.

Theorem 8. *A T_3-space $(X, \mathscr{T})$ is a metacompact Moore space iff $(X, \mathscr{T})$ satisfies E_5'.*

Note. Metacompact = pointwise paracompact = weak paracompact.

Since a Moore space has the property that every closed set is a G_δ, from Theorem 6 we can conclude,

Theorem 9. *An E_5, T_4 Moore space is metrizable. For T_4 spaces $E_5 + E_5' \leftrightarrow E_6$. Thus the example at the end of the last section is T_4 and E_5 but not E_5'.*

Bing has an example of an E_5' space that is not normal and hence not metrizable but is screenable (every open cover has a σ-disjoint open refinement). The next theorem shows this example satisfies E_5.

Theorem 10. *A screenable E_5' space $(X, \mathscr{T})$ is E_5.*

Proof. The point regular base $\mathscr{U}$ can be expressed as a countable union of point finite covers $\mathscr{U}_n$. Each point finite cover has a σ-disjoint open refinement $\mathscr{V}_n$. The family $\mathscr{V} = \bigcup \mathscr{V}_n$ is a σ-disjoint base for $(X, \mathscr{T})$.

Heath [17] proved that a screenable Moore space is metacompact and clearly an E_5 space is screenable.

Corollary 10. *For a Moore space $(X, \mathscr{T})$ the following relations hold*

(a) $\leftrightarrow$ (c) $\rightarrow$ (b) $\leftrightarrow$ (d)

(a) $(X, \mathscr{T})$ satisfies E_5.

(b) $(X, \mathscr{T})$ satisfies E_5'.

(c) $(X, \mathscr{T})$ is screenable.

(d) $(X, \mathscr{T})$ is metacompact.

Clearly the condition $E_5' \rightarrow E_5$ for T_4 spaces is equivalent to the statement every pointwise paracompact normal Moore space is metrizable. There is an example of Heath [17] which is metacompact, non-normal Moore space which is not screenable. Hence there exists a T_3, E_5', E_4 space which is not T_4 or E_5. Is there a collectionwise normal E_4 space that is not E_5?

$E_6 - E_8$ Spaces

We summarize the now classical theorems of Bing, Nagata and Smirnov along with an old theorem of Aleksandrov and Urysohn and a relative recent theorem of Arhangel'skii.

Theorem 11. *A T_3 space $(X, \mathscr{T})$ is metrizable iff*

(a) (Aleksandrov and Urysohn [5]) *it has a development $\mathscr{V} = \bigcup_n \mathscr{V}_n$ such that $\mathscr{V}_{n+1}$ is a refinement of $\mathscr{V}_n$ for all n and such that the union of each pair of intersecting elements of $\mathscr{V}_{n+1}$ is a subset of an element of $\mathscr{V}_n$.*

(b) (Nagata-Smirnov [21] and [24]) *it has a σ-locally finite base.*

(c) (Bing [11]) *It satisfies E_6.*

(d) (Arhangel'skii [6]) *there exists a base $\mathscr{V}$ for $(X, \mathscr{T})$ such that for $x \in T \in \mathscr{T}$ there exists $T' \in \mathscr{T}$, $x \in T' \subset T$ such that only a finite number of elements of $\mathscr{V}$ intersect both T' and $\sim T$.*

We need only mention the classical result of Urysohn that a topological space is metrizable and separable iff it is T_4 and satisfies E_7. Examples of E_6 spaces that are not E_7 are numerous. For instance see Thron [25, 171].

The next theorem shows that the combined properties of compactness and metrizability can be expressed in terms of a base axiom which says something about closed sets as well as points.

Theorem 12. *A T_3 space $(X, \mathscr{T})$ is metrizable and compact iff*

(a) *it satisfies E_8;*

(b) *there is a point-countable base $\mathscr{V}$ for $(X, \mathscr{T})$ such that the neighborhood system of each closed set has a base which is a subfamily of $\mathscr{V}$.*

Proof. See Aull [7] and [8].

Some Further Remarks

Ralph Root [23] had the axioms for a first countable T_2 space independent of Hausdorff. However his work is more difficult to follow and did not affect the main streams of point-set topology. See also Thron [25, 236].

Urysohn's metrization theorem about separable spaces involved T_4 spaces. Tychonoff [26] replaced T_4 by T_3.

The problem of the metrization of normal Moore spaces has an interesting history. See Heath [15] and Jones [18]. A recent review of a paper of Younglove by F. B. Jones in the July 1968 Mathematical Reviews indicates some very interesting recent developments. Finally we list some questions involving counterexamples.

1. Is there a homogeneous T_2, E_0 space that is not E_1? [1])

2. Is there a perfectly T_4, collectionwise normal E_3 space which is not E_4?

3. Is there a collectionwise T_4, E_4 space that is not E_5 or even a T_4, E_4 space that is not E_5?

Note, if every T_4, E_4 space is E_5 then every normal metacompact Moore space is metrizable.

4. Is there a T_2 space with a σ-locally finite base that does not satisfy E_6 or perhaps not even E_5?

Added to the galley. In regard to question 3, the author has proved that hereditary CN, E_4 spaces satisfy E_5. See AMS Notices 1969.

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