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HOMOMORPHISMS AND ISOMORPHISMS OF SEMIGROUPS OF CONTINUOUS SELFMAPS

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1. Introduction. It will be convenient to make the assumption that all topological spaces under consideration here are Hausdorff. Let X be any space and Y a nonempty subspace of X. The family of all continuous selfmaps of X which also take Y into Y is a semigroup under composition and is denoted by $S(X, Y)$. These semigroups were introduced in [4] and discussed further in [6]. When $Y = X$, we have the semigroup of all continuous selfmaps of X and we write $S(X)$ in place of $S(X, X)$. The problem we discuss here is that of determining when $S(Z)$ is a homomorphic image of $S(X, Y)$ and more specifically, when $S(Z)$ and $S(X, Y)$ are isomorphic.

2. S*-Spaces and Permissible Pairs. Definition 2.1. *A topological space X is an S^*-space if for every closed subset F of X and every point $p \in X - F$, there exists a continuous selfmap f of X and a point $q \in X$ such that $f(x) = q$ for $x \in F$ and $f(p) \neq q$.*

Theorems 2 and 3 of [5] combine to give

Proposition 2.2. *The class of S^*-spaces includes all completely regular spaces which contain an arc as well as all 0-dimensional spaces (those spaces which have a basic of sets which are both open and closed).*

We refer to a subset H of a topological space X as a point-inverse if $H = f^{-1}(x)$ for some $f \in S(X)$ and some $x \in X$. One easily verifies the following

Proposition 2.3. *A space is an S^*-space if and only if the family of all point-inverses is a basis for its closed subsets.*

Definition 2.4. *A permissible pair (X, Y) is a space X together with a subspace Y such that the following conditions are satisfied:*

(2.4.1) *For every closed subset F of X and every point $p \in X - F$, there exists a function $f \in S(X, Y)$ and a point $q \in Y$ such that $f(x) = q$ for $x \in F$ and $f(p) \neq q$.*

(2.4.2) *For every quadruple p, q, r, s of point of Y with $p \neq q$, there exists a continuous function f in $S(X, Y)$ such that $f(p) = r$ and $f(q) = s$.*

The next result is a straightforward consequence of condition (2.4.1).

Proposition 2.5. *Let (X, Y) be a permissible pair. Then $\{f^{-1}(y) : f \in S(X, Y), y \in Y\}$ is a basis for the closed subsets of X.*

The following two results show that permissible pairs are fairly numerous.

Proposition 2.6. *If X is any completely regular space and Y is any arcwise connected subspace containing more than one point, then the pair (X, Y) is permissible.*

Proposition 2.7. *If X is 0-dimensional and Y is any subspace with more than one point, then the pair (X, Y) is permissible.*

To prove Propositions 2.6 and 2.7 it is sufficient to show that for every closed subset F of X, $p \in X - F$ and $r, s \in Y$, there exists a function $f \in S(X, Y)$ such that $f(p) = r$ and $f(x) = s$ for $x \in F$. In the case of Proposition 2.6, there exists a homeomorphism h from the closed unit interval I into Y such that $h(0) = r$ and $h(1) = s$. By complete regularity, there exists a continuous function k from X into I such that $k(p) = 0$ and $k(x) = 1$ for $x \in F$. We can then take f to be $h \circ k$. As for Proposition 2.7, there exists a set H which is both open and closed such that $p \in H \subset X - F$. We get the desired function in this case by defining $f(x) = r$ for $x \in H$ and $f(x) = s$ for $x \in X - H$.

3. Homomorphisms and Isomorphisms. We now address ourselves to the problem of determining when $S(Z)$ is a homomorphic image of $S(X, Y)$ where Z is an S^*-space and (X, Y) is a permissible pair. The next result has by now appeared in [6, p. 137] as Theorem (4.1) and since its proof is somewhat lengthy, it will not be given here. This result reduces the previous problem to that of determining when Y is homeomorphic to Z and S-embedded in X. This leads us to

Definition 3.1. *A subspace Y of X is S-embedded in X if every continuous selfmap of Y can be extended to a continuous selfmap of X.*

Theorem 3.2. *Let (X, Y) be a permissible pair and let Z be an S^*-space. Then $S(Z)$ is a homomorphic image of $S(X, Y)$ if and only if Z consists of one point or Z is homeomorphic to Y and Y is S-embedded in X.*

As for isomorphisms, we have

Theorem 3.3. *Let (X, Y) be a permissible pair and let Z be an S^*-space. Then $S(X, Y)$ and $S(Z)$ are isomorphic if and only if Z is homeomorphic to Y and Y is a dense S-embedded subspace of X.*

Proof. First suppose that h is a homeomorphism from Z onto Y and that Y is a dense S-embedded subspace of X. The mapping which takes $f \in S(Z)$ into $h \circ f \circ$ $\circ h^{-1} \in S(Y)$ is an isomorphism from $S(Z)$ onto $S(Y)$. Furthermore, each $g \in S(Y)$ has a unique extension to a function $\varphi(g) \in S(X, Y)$. It is a straightforward matter to check that φ is an isomorphism from $S(Y)$ onto $S(X, Y)$ and it follows that $S(X, Y)$ and $S(Z)$ are isomorphic.

On the other hand, let φ be an isomorphism from $S(X, Y)$ onto $S(Z)$. We first dispense of the case where Z consists of one point. In this case, $S(Z)$ has only one element and hence $S(X, Y)$ has only one element. Thus, $X = Y$ consists of one point and it follows that Z is homeomorphic to Y and that Y is a dense S-embedded subset of X. Now consider the case where Z consists of more than one point. It follows from the previous theorem 3.2 that Z is homeomorphic to Y and Y is S-embedded in X. But equally important, an analysis of the proof (see [6, p. 137]) of that result allows us to conclude the existence of a homeomorphism h from Z onto Y such that $\varphi(f) =$ $= h \circ f \circ h^{-1}$ for each $f \in S(X, Y)$. Using this fact, we show by contradiction that Y is dense in X. Suppose $p \in X - \text{cl } Y$. Then by condition (2.4.1), there exists a function f in $S(X, Y)$ and a point $q \in Y$ such that $f(x) = q$ for $x \in \text{cl } Y$ and $f(p) \neq q$. The functions f and $\langle q \rangle$ (the constant function which maps all points of X into q) are distinct but both $\varphi(f)$ and $\varphi\langle q \rangle$ are equal to $\langle h(q) \rangle$. This contradicts the fact that φ is an isomorphism. Hence Y is dense in X and the proof is complete.

4. S-Embedded Subse.s. The two results of the previous section state that in order to determine whether or not $S(Z)$ is a homomorphic or an isomorphic image of $S(X, Y)$, one must be able to determine, among other things, if Y is S-embedded in X. In this section we investigate S-embeddedness a bit further. The first result is easily verified and its proof is omitted.

Proposition 4.1. *Every retract of a space is S-embedded in that space.*

The statements of the next two results involve realcompact spaces, the Hewitt realcompactification υX of a completely regular space X and also its Stone-Čech compactification βX. One may consult [3] for a detailed investigation of these concepts.

Proposition 4.2. *Let X be a compact space and let Y be a subspace which contains an arc. Then Y is a dense S-embedded subspace of X if and only if X is the Stone-Čech compactification of Y.*

Proposition 4.3. *Let X be a realcompact space and suppose Y is a subspace of X which contains a copy of the real line which is closed in X. Then Y is a dense S-embedded subspace of X if and only if X is the Hewitt realcompactification of Y.*

The proofs of the last two results are similar and because of this, we give the details only in the second case. First of all, if $X = \upsilon Y$, it follows from well known properties of this space that Y is dense in X and that every continuous function mapping Y into Y can be continuously extended to a function which maps X into X.

Suppose, on the other hand, that Y is dense in X and S-embedded as well. We must show that if f is any continuous real-valued function on Y, then f can be continuously extended to a real-valued function on X. By hypothesis, there exists a homeomorphism k from the real line R onto a subset H of Y which is closed in X. Then $k \circ f$ is a continuous mapping from Y into Y and since Y is S-embedded in X, $k \circ f$ has a continuous extension g which maps X into X. Now, since H is closed in X, we have

$$g[X] = g[\mathrm{cl}_X Y] \subset \mathrm{cl}_X g[Y] = \mathrm{cl}_X k \circ f[Y] \subset H .$$

That is, the range of g is a subset of H. It follows that $k^{-1} \circ g$ is a continuous extension of f which maps X into R. Thus X must be the Hewitt realcompactification of Y.

In Proposition 4.3, one cannot hope to replace the requirement that the copy of R be closed in X with the weaker requirement that the copy of R need only be closed in Y. For if we take $Y = R$, Y certainly contains a copy of R which is closed in Y and furthermore, Y is a dense S-embedded subspace of βY, its Stone-Čech compactification. However, the Hewitt realcompactification in this particular case is Y itself and not βY.

An application of a very nice result due to Dugundji [2] provides us with other examples of S-embedded subspaces. First we state Dugundji's theorem.

Theorem 4.4 (Dugundji). *If f is any continuous function mapping a closed subset A of a metric space X into a locally convex topological linear space L, then there exists a continuous extension of f which maps X into the convex hull of $f[A]$.*

The following fact is an immediate consequence of Dugundji's theorem.

Proposition 4.5. *Let X be any convex subset of a normed linear space. Then any closed (in the topological sense) subset of X is S-embedded in X.*

However, being closed and being S-embedded are not equivalent. DeGroot [1] has proven the existence of 2^c one-dimensional connected, locally connected subspaces of the Euclidean plane E^2 with the very interesting property that for any two such spaces, only constant functions map one continuously into the other and the only continuous selfmaps of any such space are the constant functions and the identity function. Let Z be any such space and let $Y = h[Z]$ where h is any homeomorphism from E^2 onto the interior of the closed unit disk X in E^2. Since the only continuous selfmaps of Y are the constant functions and the identity function, it is immediate that Y is S-embedded in X which is a convex subset of a normed linear space. However, Y is not a closed subset of X for an assumption to the contrary leads one to the conclusion that Y is compact, connected, locally connected and metric, i.e., a Peano space with more than one point. But these spaces have many continuous selfmaps in addition to the constant maps and the identity map. Consequently, Y cannot be closed in X.

For convex subsets of a convex set X, however, being closed and being S-embedded are equivalent. We state this formally as

Proposition 4.6. *Let X be a convex subset of a normed linear space and let Y be a convex subset of X. Then Y is S-embedded in X, if and only if Y is closed (in the topological sense) in X.*

Proof. If Y is closed, it follows immediately from Proposition 4.5 that Y is S-embedded in X. Suppose Y is not closed. Since X is a first countable space, there exists a point $p \in X - Y$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points of Y which converges to p. Let $A = \{x_n\}_{n=1}^{\infty}$ and define a function f mapping A into A by $f(x_{2n}) = x_{2n}$ and $f(x_{2n-1}) = x_1$. The function f is continuous since A is discrete. Furthermore, A is a closed subset of Y and, hence, by Dugundji's theorem, f has a continuous extension to a function g which maps Y into Y. However, g cannot possibly be extended continuously over X since $\lim x_n = p$ while $\lim f(x_n)$ does not exist.

Now we combine the two results of section 3 with these last several results on S-embeddedness. In all the following statements, (X, Y) is a permissible pair and Z is an S^*-space.

Theorem 4.7. *Suppose X is compact and Y contains an arc. Then $S(X, Y)$ is isomorphic to $S(Z)$ if and only if Z is homeomorphic to Y and X is the Stone-Čech compactification of Y.*

Theorem 4.8. *Suppose X is realcompact and Y contains a copy of the real line which is closed in X. Then $S(X, Y)$ is isomorphic to $S(Z)$ if and only if Z is homeomorphic to Y and X is the Hewitt real compactification of Y.*

Theorem 4.9. *Suppose X is a convex subset of a normed linear space and Y is a convex subset of X. Then $S(Z)$ is a homomorphic image of $S(X, Y)$ if and only if Z consists of one point or Z is homeomorphic to Y and Y is a closed subset of X.*

Theorems 4.7 and 4.8 follow from Theorem 3.3 combined respectively with Propositions 4.2 and 4.3. Theorem 4.9 is a consequence of Theorem 3.2 and Proposition 4.6. A result which is similar to Theorem 4.7 appears in [4] but the techniques which were used there are not the same as those used here and different assumptions were made on the spaces involved. We remark in closing that in Theorem 4.9, if Y has more than one point, Proposition 2.6 implies that (X, Y) is permissible. For X is certainly completely regular and Y is arcwise connected since it is a convex subset of X. Theorem 4.9 now appears in [6, p. 140] as Corollary (4.3).

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