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Top and its adjoint relatives

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0. Introduction

This paper is an informal presentation of much of Chapter III of "General functorial semantics" (to be published) including substantial outlines of some of the proofs. It was the basis of my lectures at the Topological Conference in Kanpur which treated more examples. I hope it may be intelligible to many readers, either by itself or together with the much longer forthcoming paper.

1. Connections and Topological Objects

Let us recall what adjointness means — especially because it comes in four forms, one of which is of primary interest here.

We are concerned with two categories $\mathscr{C}$, $\mathscr{D}$ and two contravariant functors C: $\mathscr{C} \rightsquigarrow \mathscr{D}$, H: $\mathscr{D} \rightsquigarrow \mathscr{C}$. An *adjunction on the right* α of C and H is a natural equivalence between two functors $h_1, h_2 : C^{op} \times D^{op} \to S$ symbolized by $h_1(X, A) = \mathscr{C}(X, HA)$, $h_2(X, A) = \mathscr{D}(A, CX)$. This notation omits some necessary details — for f: $X \to X'$, g: $A \to A'$, there is $h_1(f, g)$ taking morphisms m: $X' \to HA'$ to the composite (Hg) mf: $X \to HA$, and similarly for h_2. Some further such details will be usually omitted here. In particular, if there is an adjunction of C and H, there may be other adjunctions of the same functors. Fortunately, all of them are very much alike, and we may usually write "$\mathscr{C}(X, HA) \sim \mathscr{D}(A, CX)$" and treat "$\sim$" like an equality.

It will be convenient to speak of the pairs of functors C, H, adjoint on the right, as *connections*. (Suitably restricted ones will be called *Galois connections*.) The first thing about connections is that, in the case of interest here, $\mathscr{C} = \mathbf{Top}$, they are substantially the same thing as topological objects in $\mathscr{D}$: objects of $\mathscr{D}$ provided with a compatible topology.

To see this, let us go back to the simpler notion of a *group* in $\mathscr{D}$. A group in $\mathscr{D}$ may be described as an ordered quadruple $(A, ., ^{-1}, e)$ forming a diagram

$$A^2 \xrightarrow{\ \cdot\ } A \circlearrowleft^{-1} \xleftarrow{\ e\ } A^0$$

in $\mathscr{D}$, satisfying the usual laws for groups. For instance, the associative law $[xy]\,z = x[yz]$ asserts the equality of two morphisms $\gamma_1, \gamma_2 : A^3 \to A$; γ_1 runs $(x, y, z) \to ([xy], z) \to [[xy]\,z]$, and the γ_j can be described in terms of product objects and without reference to elements x, y, z.

This is a diagram in $\mathscr{D}$, on an index category with three objects 0, 1, 2, and suitable morphisms, which takes each index object i to A^i for some A and satisfies additional conditions that may (like associativity) involve A^3. Lawvere has shown [L] that a general theory of algebras in a category is possible (and manageable) as follows. One uses another index object 3, and in fact i for every natural number i; one adjoins morphisms between index objects a such as $\pi_1 : 2 \to 1$, $\pi_2 : 2 \to 1$, so that i is already the i-th power of 1 in the index category; and one finds finally that an algebra in $\mathscr{D}$ is nothing but a product-preserving functor $\mathscr{I} \to \mathscr{D}$ on the amplified index category $\mathscr{I}$.

For any sort of algebra, the appropriate index category turns out to be just the dual of the category of finitely generated free algebras of that sort. A great deal can be done with this system of (*algebraic*) *functorial semantics*, which we must slide over here in order to get to the notion of a topological object in $\mathscr{D}$. Just note now that in introducing the indices 3, 4, ..., one has lost a little: there cannot now be a group in $\mathscr{D}$ except on an object A all of whose finite powers exist in $\mathscr{D}$. Most of the categories of sets with a type of structure, in which one wants group objects or topological objects, are *complete* categories, having products and (inverse) limits of arbitrary diagrams. Most of them also satisfy more technical conditions (such as having a "generator") which will be alluded to here as "being well-behaved".

When $\mathscr{D}$ is complete, any product-preserving functor A: $\mathscr{I} \to \mathscr{D}$ on the dual $\mathscr{I}$ of the category of finitely generated free algebras of a given sort admits an extension A over the dual $\mathscr{V}^{op}$ of the variety of all algebras of that sort, preserving all limits in $\mathscr{V}^{op}$. Moreover, every object of $\mathscr{V}^{op}$ is a limit of objects of $\mathscr{I}$, this is how A can be constructed – and it follows that A is unique up to an isomorphism of functors (natural equivalence). The category of algebras in $\mathscr{D}$ is equivalent to the category of contravariant functors C: $\mathscr{V} \to \mathscr{D}$ which take all colimits (direct limits) in $\mathscr{V}$ to limits in $\mathscr{B}$.

Every variety $\mathscr{V}$ is "well-behaved"; each of these functors C has an adjoint on the right, and thus yields a connection (C, H) between $\mathscr{V}$ and $\mathscr{D}$. Conversely, if (C, H) is a connection, C takes all colimits to limits. Of course C does not strictly determine H, but it determines H up to isomorphism. Selecting adjoints HC for all $\mathscr{V}$-algebras C in $\mathscr{D}$ (which is not greatly different from selecting a square A^2 of an object A) and selecting adjunctions (cf. coordinate projections), natural transformations $C \to C'$ correspond biuniquely to natural transformations $H \to H'$. Omitting some details of definition, we conclude:

The category of $\mathscr{V}$-algebras in $\mathscr{D}$ is equivalent to the category of connections between $\mathscr{V}$ and $\mathscr{D}$.

Remark. This is true for arbitrary $\mathscr{D}$ if we take $\mathscr{V}$-algebras to be suitable functors $\mathscr{V}^{op} \to \mathscr{D}$; true for complete $\mathscr{D}$, with the usual definition of $\mathscr{V}$-algebras $\mathscr{I} \to \mathscr{D}$.

Accordingly we are on firm ground if we define a topological object in a variety $\mathscr{V}$ as a functor H: $\mathbf{Top}^{op} \to \mathscr{V}$ preserving all limits. In fact we are on firm ground in defining a topological object in any category $\mathscr{D}$ as a limit-preserving functor C: $\mathbf{Top}^{op} \to \mathscr{D}$; and we do so. The ground is general functorial semantics, which has not been developed in the literature (though considerable fragments of it are in Benabou's thesis [B1]). It cannot be developed in this lecture; we allude to it in justification of calling a connection of **Top** and $\mathscr{D}$ a topological object.

2. Topological $\mathscr{D}$-sets

Topological groups are older than functors, and are usually defined as sets G with two structures making G a topological space and a group in a compatible way. This involves not only the categories **Top** and **Gr** of topological spaces and groups but also the standard forgetful functors U: $\mathbf{Top} \to \mathscr{S}$, U': $\mathbf{Gr} \to \mathscr{S}$. Note that we already have the category $\mathbf{Top} \otimes \mathbf{Gr}$ of topological groups, without forgetful functors. Introducing U and U', which can be varied considerably, we get forgetful functors

(1)
$$\begin{array}{ccc} \mathbf{Top} \otimes \mathbf{Gr} & \longrightarrow & \mathbf{Top} \\ \downarrow & & \downarrow \\ \mathbf{Gr} & \longrightarrow & \mathscr{S} \end{array}$$

It must be noted that these matters are more special than the preceding ones; now think of categories $\mathscr{D}$ that are definitely well-behaved, like **Top** or a variety. A forgetful functor in general is required to preserve limits. Hence a forgetful functor on $\mathscr{D}$ (which is well-behaved) is representable. The *types* $\mathscr{D}$ of set with structure in which we are interested are given by (such) a category $\mathscr{D}$ with a distinguished object P.

For types $\mathscr{D}_1$, $\mathscr{D}_2$, we may define $\mathscr{D}_1 \otimes \mathscr{D}_2$ (simply) as the category of $\mathscr{D}_1$-objects C in $\mathscr{D}_2$. The forgetful functor $\mathscr{D}_1 \otimes \mathscr{D}_2 \to \mathscr{D}_2$ takes C to $C(P_1)$. For $\mathscr{D}_1 \otimes \otimes \mathscr{D}_2 \to \mathscr{D}_1$, form the composite $U_2C : \mathscr{D}_1 \rightsquigarrow S$; this is a $\mathscr{D}_1$-object in $\mathscr{S}$, thus representable, and one gets a forgetful functor by choosing representations. The diagram (1) is not strictly commutative; the composite $C \mapsto U_2\, C(P_1)$ and the other composite are merely isomorphic functors. The distinguished object P is a representing object for these functors, which exists because the functors preserve limits and $\mathscr{D}_1 \otimes \mathscr{D}_2$ is well-behaved.

The distinguished objects P_1, P_2 are arbitrary, P depends on them by a functor $\otimes$: $\mathscr{D}_1 \times \mathscr{D}_2 \to \mathscr{D}_1 \otimes \mathscr{D}_2$, the tensor multiplication of objects.

Holding P_1 fixed, the functor $P_1 \otimes \_ : \mathscr{D}_2 \to \mathscr{D}_1 \otimes \mathscr{D}_2$, is left adjoint to the forgetful functor $C \mapsto C(P_1)$. [Remark: as "$\otimes$" is adjoint to "Hom" in abelian groups. Evaluation $C(P_1)$ is related to homming by the Yoneda lemma. This is not a set-valued Hom but a lifted one, just as in abelian groups. Roughly, $\otimes$ is adjoint to Hom in general.]

Since (1) is nearly commutative, it gives us for each $\mathscr{D}_1 \otimes \mathscr{D}_2$-set M an underlying $\mathscr{D}_1$-set M_1 and an underlying $\mathscr{D}_2$-set M_2 on isomorphic underlying sets. For a topological group M one expects something sharper: a space M_1 and a group M_2 on exactly the same underlying set. Moreover, this M_1 and M_2 determine M.

(M_1 and M_2, given up to isomorphism, do not determine M up to isomorphism. For that, take two connected groups A, B, that are homeomorphic not isomorphic, such as the translations of the plane and the order-preserving affine transformations of the line. If A', B', are the corresponding discrete groups, then $A \times B'$ and $A' \times B$ establish the point.)

To discuss this in general one needs a more detailed picture than (1). Note that (1) should really include a natural equivalence α between the two composites. This makes it a 2-dimensional diagram with a 2-cell α ("higher dimensional abstract nonsense": a necessary study, systematically begun in [B2]). We need do no more than recall how we got (1).

A $\mathscr{D}_1 \otimes \mathscr{D}_2$-set C was defined as a $\mathscr{D}_1$-object C: $\mathscr{D}_1^{op} \to \mathscr{D}_2$. Then U_2C is a $\mathscr{D}_1$-set on the set $S = U_2\, C(P_1)$, and $C(P_1)$ is an object of $\mathscr{D}_2$ on the same set S. So stated, this is for arbitrary $\mathscr{D}_j$.

If P_2 is a generator in the sense of Freyd and P_1 is a generator (in the sense of Grothendieck) then U_2C and $C(P_1)$ determine C up to a unique isomorphism.

This will require some explanation. *Freyd* calls P a *generator* in C if the covariant functor h^P represented by P is faithful, i.e. for any two morphisms $f, g: X \to Y$ in $\mathscr{C}$, for some e: $P \to X$, $fe \neq ge$[F]. The form of Grothendieck's definition [G]: for every object X and proper subobject W of X, there is e: $P \to X$ not factoring through W. We shall not need to define *subobject*, for in wellbehaved categories, P is a *generator* if and only if no proper subclass of the class of objects includes P and is closed under the formation of colimits [I3]. Thus every object can be constructed from P by transfinitely iterated formation of colimits. This implies that P is a Freyd generator (or *coseparator*).

It is not enough to have two Freyd generators P_j. To see this, consider complete lattices $\mathscr{B}$. Every object is a Freyd generator; but taking $P_j = 0$ (the least element), both $C(P_1)$ (the greatest element) and U_2C are essentially the same for all C. But not all C are isomorphic.

To amplify "up to a unique isomorphism": if $U_2C = U_2C'$ and $C(P_1) = C'(P_1)$, then there exist isomorphisms $\alpha : C \to C'$, and there is just one such that $U_2\alpha$ and αp_1 are identities. Moreover, any (iso-)morphisms α_2, α_1, between the composites with U_2 and the values at P_1, which induce the same (iso-)morphism of underlying sets, are induced by a unique α; but the proof of this will be omitted.

The proof of the first part will be condensed. Using transfinite induction, one defines α_X: $C(X) \to C'(X)$ for a colimit X of objects where α was previously defined, by choosing a representation of X as a colimit of (say) W's. C and C' take this to representations as a limit. The morphisms α_W will induce a (unique) morphism α_X of the limits if they commute with the morphisms in the diagrams, and thus if the portion of α that was previously defined is natural. To prove that (inductively) one proves first $U_2\alpha = 1$. We began with $\alpha_{p_1} = 1$. If all $U_2(\alpha_W)$ are identities, then the coordinates of $U_2(\alpha_X)$ in the $U_2\, C'(W)$ $(=U_2\, C(W))$ are the coordinates of the identity, so $U_2(\alpha_X) = 1$. Then α is natural because $U_2\alpha$ is natural, and U_2 is faithful.

As for uniqueness, there is at most one natural transformation α: $C \to C'$ such that either $U_2\alpha$ or α_{p_1} is an identity.

From this theorem and the essential symmetry of (1) it follows that something very similar must be true if P_2 is a genuine generator and P_1 a Freyd generator. In fact, exactly the same result holds then, and the proof transforms in a simple way. This adds some light on techniques, and also reduces the good behaviour hypotheses (which are considerably less for these proofs than for (1)). Here is a sketch. Fix X in $\mathscr{D}_1$. One has 1: $U_2\, C(X) \to U_2\, C'(X)$, or by representing U_2, α_X^0: $\mathrm{Hom}\,(P_2, C(X)) \to$ $\to \mathrm{Hom}\,(P_2, C'(X))$. This gives a natural transformation between the corresponding functors on the full subcategory $\mathscr{A}^o$ of $\mathscr{D}_2$ on the one object P_2: natural, because it is natural after (faithful) homming into $C(P_1)$. The induction runs as before up subcategories A_β of more and more colimits; when A_β takes in $C(X)$, one has α_X. Finally, α is natural because of homming into $C(P_1)$.

3. Topological $\mathscr{D}$-sets, continued

To describe a group M in a suitable concrete category $\mathscr{D}_2$ one has an underlying object M_2 and an underlying group M_1, on the same ground set. It can happen that M_1 and M_2 do not suffice, though one usually does have faithful forgetful functors. Even then, M is no more than a quite small diagram in $\mathscr{D}_2$ around M_2: M_2 and its multiplication table.

For topological objects M in sufficiently suitable $\mathscr{D}_2$, one has again a sufficient description by means of the object M_2 and the topological space M_1. The theorem does not apply to topological spaces in **Top**, for instance, and we shall see that it is false for them.

A connection between **Top** and **Top** consists of adjoint functors C, H. UC is represented by a space $V = H(P)$ (by adjointness). The space $V' = C(P)$ is on $U\, C(P) = \mathrm{Hom}\,(P, V) = U(V)$. Discrete spaces S are taken by C to $(V')^S$: coproduct to product. So any X, as a quotient of the discrete space on the same set, goes to a topological space $C(X)$ on $\mathrm{Hom}\,(X, V)$, mapping continuously (by identification of V' with V) into the product $(V')^X$. Thus (i) $C(X)$ is the set of functions $\mathrm{Hom}\,(X, V)$

in a topology finer than the topology of pointwise V'-convergence. (ii) C is functorial, i.e. the functions $U\,C(f)$ are continuous. This can be broken up into three simpler statements, according to the factorization of f into a quotient map f_0, a coarsening f_1, and an embedding f_2; moreover, (iii) $C(f_0)$ must be a topological embedding, since f_0 is a coequalizer and C dually preserves colimits. And (iv) C dually preserves coproducts (disjoint sums).

There are no further conditions; **Top** is well-behaved, and every C satisfying (i)–(iv) has an adjoint. There is no relation between V and V' beyond the common underlying set. For any V and V' on the same set, one can give all $C(X)$ the topology of pointwise-convergence. The adjoint H then is also defined by pointwise convergence (with the roles of V and V' reversed).

There are other connections between **Top** and **Top**. The original version of this paper described one example. It turns out that besides the coarsest topology (pointwise V'-convergence as in (i) above) there is a finest. In it, the limits of a filter Λ of continuous functions $X \to V$ are those V'-pointwise limits g such that X has an open covering by sets U_i on each of which either (1) g and all elements of some $L_i \in \Lambda$ are constant, or (2) all elements of some $L_i \in \Lambda$ coincide with g. The proof will appear in "General functorial semantics".

It is not known whether the topology just described or anything different from the pointwise topology, yields a connection of the category of Tychonoff spaces with itself.

None of the usual topologies for function spaces such as the compact open can establish a connection. It is easy to satisfy (i) and (ii). However, the further conditions imply that *C is determined up to isomorphism by its effect on Hausdorff ultraspaces*. For the Hausdorff ultraspaces form a left adequate subcategory of **Top** ([I2] proof of 9.1).

This incomplete description of **Top** $\otimes$ **Top** contains something remarkable: any V and V' on the same set can be used. In other words, the functor **Top** $\otimes$ **Top** $\to$ $\to$ **Top** $\underset{\mathscr{S}}{\times}$ **Top** implicit in diagram (1) is surjective on objects. We actually have rather more: a uniform way of using V and V', making a right inverse **Top** $\underset{\mathscr{S}}{\times}$ **Top** $\to$ $\to$ **Top** $\otimes$ **Top**. More can be said about this in connection with other natural functors in the system, but let us stop at the surjection. This generalizes, not very far, but far enough to apply to T_1-spaces, partially ordered sets, and some other categories sufficiently like **Top**.

Consider well-behaved $\mathscr{D}_1$, $\mathscr{D}_2$. To justify a simplified statement we use a weak *transportability* assumption on $\mathscr{D}_2$: for any object X of $\mathscr{D}_2$ and bijection $f\colon U_2X \to S$ there exist X' in $\mathscr{D}_2$ and an isomorphism $\Phi\colon X \to X'$ such that $U_2(\Phi) = f$.

Given a representable functor $W\colon \mathscr{D}_1^{op} \to S$ and an object V of $\mathscr{D}_2$ such that $U_2(V) = W(P_1)$: to lift W to a functor $C\colon \mathscr{D}_1^{op} \to \mathscr{D}^2$, dually preserving colimits, such that $U_2C = W$ and $C(P_1^2) = V$. If (a) P_1 is a Freyd generator, (b) every morphism from P_1 to a copower of P_1 is a coordinate injection, and (c) for every object Y of $\mathscr{D}_2$

and subset S of $U_2(Y)$ the subfunctor hS of h_Y whose values $h\,S(X)$ consist of all f: $X \to Y$ such that $U_2(f)$ factors through S is representable, then C exists. It can be constructed in a natural way, so that (though the proof will be omitted) for two such problems given by W, V, and W', V', any morphism $W \to W'$ and $V \to V'$ agreeing on $W(P_1)$ are induced by a morphism $C \to C'$.

For the construction, of course the I-th copower $I.\,P_1$ must go to an I-th power V^I; by transportability, there is an I-th power object properly situated on the underlying set $W(I\,.\,P_1)$. Because of (b), this effects the lifting for the full subcategory of copowers of P_1. Any other object K of $\mathscr{D}_1$ is a quotient of the ("discrete") copower $U_1(K)\,.\,P_1 = |K|$; accordingly $W(K)$ is a subset S of $W(|K|)$, and we construct $C(K)$ in $C(|K|)$ by means of (c). This takes care of all the morphisms from P_1 to K, coordinates of $|K| \to K$ which one maps to coordinates of $C(K) \to C(|K|)$. Morphisms from a copower $I\,.\,P_k$ to K are described by their coordinates $P_1 \to K$ and mapped accordingly. For any $K \to K'$, we have C defined on the composite $|K| \to K \to K'$, which maps to $C(K') \to V^{|K|}$; since U_2 takes this to a map factoring through $W(K)$, it factors uniquely through $U(K)$.

Finally, C is a functor because equality of morphisms into $C(K)$ reduces to equality of their composites with $C(K) \to C(|K|) \to V$. $U_2C = W$ dually preserves colimits, and we have lifted the limits from S to $\mathscr{D}_2$.

Summarizing, to describe a topological $\mathscr{D}$-set it often suffices ($\mathscr{D}$ well-behaved and P generating it) to specify the underlying $\mathscr{D}$-object and topological space, on the same ground set. At worst ($\mathscr{D}$ still well-behaved) one needs the object $C(P)$ of $\mathscr{D}$ and the subobjects $C(K)$ of powers $C(|K|)$ of $C(P)$ for all Hausdorff ultraspaces K: the objects of convergent K-tuples. The conditions for such a description to describe a topological D-set are less visible. But they tend to vanish if $\mathscr{D}$ has all imaginable subobjects ("tend" only, because the theorem above treats restricted $C(K)$'s with "pointwise" $\mathscr{D}$-structure). One has the essence of the condition if one says that all the subsets Hom (K, V) of powers $C(P)^{|K|}$ must be subobjects.

4. Universal and Galois Connections

For an abstract category $\mathscr{D}$, a *universal* connection is a pair of dual isomorphisms (F, F^{-1}) connecting $\mathscr{D}$ with $\mathscr{D}^{op}$. (These are adjoint on the right as well as mutually inverse, and adjoint on the left.) Evidently (F, F^{-1}) is a distinguished connection, and it must be universal in some sense. The sense intended here will come out when we look at concrete categories.

A *Galois* connection of abstract categories is a connection (C, H) such that CHC is naturally equivalent to C and HCH is naturally equivalent to H. Every Galois connection is an inflated universal connection, in the following sense. Given Galois (C, H) between $\mathscr{D}_1$ and $\mathscr{D}_2$, there exist an isomorphic connection (C', H'), full embeddings I_j: $\mathscr{A}_j \to \mathscr{D}_j$ a universal connection (F, F^{-1}) between $\mathscr{A}_1$ and $\mathscr{A}_2$, and reflectors R_j: $\mathscr{D}_j \to \mathscr{A}_j$ such that $C' = I_2FR_1$ and $H' = I_1F^{-1}R_2$. The proof is elementary in the technical sense and straightforward, and we do not have time for it.

An abstract category has "all possible" universal and Galois connections, and there doesn't seem to be much interest in them until we move to concrete categories. But two topics may be mentioned at this level; one of them might be very interesting, if properly understood.

What sort of object in $\mathscr{D} \otimes \mathscr{D}^{op}$ is the universal connection? In other words, in the category of endofunctors of $\mathscr{C}$ $(=\mathscr{D}^{op})$ having left adjoints, what sort of object is the identity? On the elementary level, the answer seems to be "not remarkable"; that is, the algebra of adjunction says nothing about morphisms between general right adjoints and 1. So I prepared that answer, intending to remark that nevertheless universal connections of concrete interest, such as Pontrjagin duality and Stone duality, tend also to be objects of special interest in the category of connections. But they seem to tend to be objects of the same sort of special interest: cogenerators. Pontrjagin duality connects the categories **Ab** of abelian groups and **Cab** of compact (Hausdorff) abelian groups. **Ab** $\otimes$ **Cab** is again **Cab** (in fact, with the usual forgetful functors, the forgetful functor from the tensor product to the second factor is an equivalence). The Pontrjagin object is a circle, a cogenerator. As for Stone duality, it connects the Boolean spaces **Bo** and the Boolean algebras **Ba**. That it is a cogenerator is a non trivial result of Banaschewski (communicated at this conference); but it is the reverse of surprising. In the concrete formulation, the Stone duality depends utterly on the facts that the underlying objects, the two-point space and two-element algebra, are cogenerators.

One need not go far to find that the universal connection may fail to cogenerate $\mathscr{D} \otimes \mathscr{D}^{op}$ but it can still fail gracefully. Consider the variety $\mathscr{M}$ of monoids. $\mathscr{M}$ has an involution σ, where $\sigma(X)$ has the same ground set as X but the opposite multiplication, and $\sigma(f)$ has the same values as f. Then the universal connection $\Omega = (F, F^{-1})$ and the connection $\Sigma = (F_\sigma, \sigma F^{-1})$ have only zero morphisms to each other. Since $\Sigma \times \Sigma$ has two different coordinate projections, Ω is no cogenerator.

What obstructs Ω in $\mathscr{M} \otimes \mathscr{M}^{op}$ is (at least) the *automorphism class* group F of $\mathscr{M}$, which is Z_2. If one drops down to **Gr**, σ and 1 become isomorphic, killing the automorphism class group. Σ and Ω become isomorphic. I do not know if Ω cogenerates in **Gr** $\otimes$ **Gr**^{op}. In "General functorial semantics" there is an example, the category of componentwise separable real analytic curves with boundary (based on a paper of M. Stanley to appear in J. Australian Math. Soc.), whose automorphism class group vanishes but whose universal connection is not cogenerating. It is not clear whether this can be profitably explained by "higher obstructions". I do not know a well-behaved category with trivial automorphism class group whose universal connection is not cogenerating. It seems likely that they exist; but it also seems likely that there is more structure in the question.

From just these examples one might hope that Ω is a cogenerator if $\mathscr{D}$ has a generator G and a cogenerator C. Not so, even with $G = C$. Use the based sets having two kinds of point besides the base point, and functions preserving kind or taking anything to the base point. For G, three points suffice. The σ, Σ argument fits here too.

The second topic that seems to belong to the abstract level is, how many Galois connections are there between a given $\mathscr{D}_1$ and $\mathscr{D}_2$? Let us just note that "one" is a likely answer. There is always a zero connection, given by the duality between the trivial subcategories on the product of the empty family of objects ("final object" or "right zero"). If $\mathscr{D}_1 = \mathbf{Top}$ the empty product P must go to an object of $\mathscr{D}_2$ with the same trivial monoid of endomorphisms. For instance, if $\mathscr{D}_2 = \mathbf{Gr}$, there is only the zero group, and this determines the connection. If $\mathscr{D}_2 = \mathbf{Top}$, there is another trivial possibility ("empty connection"). Note that no such argument applies between **Gr** and **Gr**. The remarkable feature of **Top** is that the empty product P is a Freyd generator, so that under side conditions a connection can be determined by its effect on P. (In $\mathbf{Gr} \otimes \mathbf{Gr}$, which is **Ab**, there appear to be again no non-zero Galois objects; but for much more sophisticated reasons, and I have not checked it.)

This is the place to notice a great many Galois connections between **Top** and a curious variety $\mathscr{V}$, consisting of the algebras with $2^{\aleph_0}$ 0-ary operations (constants) c_α and no other operations. Each of the connections we are speaking of takes P to a free algebra V_2 on 0 generators, which is just the set of constants. To determine the connection (since V has a generator and **Top** a Freyd generator) it remains to specify the topological space V_1 on the same ground set as V_2 which is to correspond to the free algebra P_2 on one generator. Any space on that ground set will yield a connection; our theorem to that effect does not apply, but it is easy to construct a connection nevertheless. A Galois connection is not so easy, but it is easy to see that it is sufficient that V_1 should have no continuous maps into itself except the identity and the constants. This implies that V_1 is connected, and determines the (trivial) continuous maps from any power of V_1 to V_1. The resulting connection is Galois, resting on the universal connection between the category of powers of V_1 and the empty space and the category of free algebras and the empty product. The required spaces V_1 have been constructed in wholesale lots, separable metric examples by de Groot [dG], compact metric examples by H. Cook [C].

Remark. De Groot and Cook have large families of these spaces which also have no continuous maps into each other except the constants. Some special interest attaches to the question whether there is such a family that is "large" in the technical sense, i.e. a proper class. The products of spaces in such a family, and the empty space, would form a full subcategory of **Top** that is closed under formation of limits but not reflective.

Let us conclude by considering universal connections and the derivation of other, possibly useful connections from them, particularly apropos of **Top** and the real line. For any connection (C, H) between concrete categories $(\mathscr{D}_1, P_1)$ and $(\mathscr{D}_2, P_2)$, we shall say that the objects $V_1 = H(P_2)$ and $V_2 = C(P_1)$ *represent* the connection. (We have seen that they need not determine it, but they do their best.) A universal connection between $(\mathscr{D}, P)$ and $(\mathscr{D}^{op}, W^{op})$, represented by W and P^{op}, is universal among connections of $(\mathscr{D}, P)$ represented by W and something, in a sense which is readily worked out. For arbitrary $\mathscr{D}$, every such connection with $(\mathscr{E}, Q)$ is associated with a colimit – preserving covariant functor F: $\mathscr{E} \rightsquigarrow \mathscr{D} \rightsquigarrow \mathscr{D}^{op}$ taking Q to W^{op}. F determines the connection up to isomorphism; and for well-behaved $\mathscr{E}$ every such F induces a connection, represented by W and something.

In general functorial semantics, a (well-behaved) category $\mathscr{E}$ with a distinguished object Q is a (complete) *theory*, and a functor preserving colimits and distinguished objects is a *morphism of theories*. This generalizes what one gets from ordinary interpretations, such as the obvious interpretation of the theory of semigroups in the theory of groups or the theory of rings, or the interpretation of the theory of Lie rings in the theory of associative rings by means of the bracket multiplication. The passage from interpretations to theory morphisms is not generally reversible; the preliminary work of "understanding" a theory is illustrated by our reduction of a topological space in $\mathscr{D}$ to a ground object A and the objects of K-convergent $|K|$-tuples in A.

In particular, $(\mathbf{Top}^{op}, R^{op})$ (connected universally to **Top**) is the universal theory of connections of **Top** represented by R. Without describing the total structure of the Lawvere theory of commutative unitary rings, we may note that to fix an interpretation of it in $(\mathbf{Top}^{op}, R^{op})$ up to isomorphism it suffices to fix the interpretations of the constants 0, 1, and the operations $+, -, .$ (Of course, consistently with the commutative law and the other axioms.) The standard interpretation takes 0 and 1 to the corresponding morphisms $R^{op} \to \Phi \,.\, R^{op}$ (dual to $R^0 \to R$) and $+, -,$ and . to the corresponding morphisms $R^{op} \to 2R^{op}$. It determines up to isomorphism an interpretation of the complete theory of commutative unitary rings in $\mathbf{Top}^{op}$, R^{op}, and the standard Galois connection between **Top** and **Cur**.

Substantially the only way I know to prove that a connection is Galois is to exhibit the universal connection on which it rests. Here, Hewitt in effect defined Q-spaces [H] as spaces X naturally homeomorphic with $H\,C(X)$, and showed that this C on completely regular spaces factors across a reflector upon Q-spaces. It is easy to verify that the image of C in **Cur** is full and reflective [I1], and trivial to extend over **Top**.

This Galois connection between **Top** and **Cur** wipes out a large part of the structure of **Top**; accordingly, it is usually considered as belonging to completely regular or Q-spaces. It also wipes out much of the structure of **Cur**. Unfortunately, **Cur** has no proper subvariety, in the classical sense, which contains the image of C. But there are a number of other varieties connected to **Top** by essentially the same Galois connections [HIJ], [HJ], [I1], some of which are even full reflective in **Cur**. From the point of view of functorial semantics, these are subvarieties. One has a subvariety whenever a morphism of algebraic theories $\mathscr{T}_1 \to \mathscr{T}_2$ induces a full embedding (necessarily reflective) $\mathscr{V}_2 \to \mathscr{V}_1$. I think that no explicit characterization of these morphisms of theories is known. (An example, hopefully typical, is the obvious interpretation of the theory of monoids as a part of the theory of groups; groups have an additional operation $^{-1}$, but its values are determined by the values of the other operations.) So we are interested in algebraic theories $\mathscr{T}$ stronger than the theory of **Cur**, such that the given interpretation of algebraic **Cur** in $(\mathbf{Top}^{op}, R)$ factors across $\mathscr{T}$.

Every complete theory $\mathscr{U}$ has an algebraic part $\mathscr{U}_a$, universal for interpretations of algebraic theories in $\mathscr{U}$. The construction is trivial; in this case $\mathscr{U} = (\mathbf{Top}^{op}, R^{op})$, $\mathscr{U}_a$ is the dual of the full subcategory of **Top** on the finite powers of R (of course, with distinguished object R^{op}). The corresponding universal variety V connected to **Top** by R is not entirely new; it contains the rings of real-valued functions closed under all continuous n-ary operations [I1], and certain archimedean lattice-ordered rings of generalized functions [HJ]. But much of the point of V is in the fields in it, of which only the subfields of R are archimedean. (Of course all these algebras are lattice-ordered, $\vee$ and $\wedge$ being continuous binary operations.) Something is known about the fields that are homomorphic images of rings $C(X)$ [EGH]; very little about fields which merely admit continuous operations beyond $\vee$ and $\wedge$ [A]. Still, however unfamiliar or difficult the algebra of V might be, it seems that if one wishes to study arbitrary continuous functions, one should be prepared to use arbitrary continuous operations.

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