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The question of whether every Moore space can be (densely) embedded in a Moore space which is Moore-complete has been answered in the negative by Mary Ellen (Estill) Rudin in [E]. The proof involves a second concept of completeness for Moore spaces, called Rudin completeness in [AL1], and it is based on the fact that there are Rudin-complete spaces which are not Moore-complete. The example [Theorem 9 of [E]] of such a space has been highly useful in the study of completeness properties [AL1][CCN][L][M][WW1][WW2].

The purpose of this paper is to simplify the example of [E] in such a way that its basic properties are preserved. The construction has been inspired by the "space Ψ" of Isbell, a description of which can be found in [GJ]. We adopt the notation of [AL1].

Example.

For every real number x we take a maximal family C_x of sequences in $\mathbb{R}\setminus\{x\}$ converging to x, such that the intersection of any two is finite.

Let $X = (\mathbb{R} \times]0,1]) \cup (\bigcup_{x\in\mathbb{R}} C_x)$. A topology on X is defined as follows. For $x \in \mathbb{R}$, $c \in C_x$ with $c = \{c_i \mid i \in \mathbb{N}\}$ and for $n \in \mathbb{N}$ let $V_n(c) = \{c\} \cup \{(y,z) \mid z \in]0,\frac{1}{n}]$ and $y = c_i$ for some $i \geq n\}$.

We take $(V_n(c))_{n\in\mathbb{N}}$ as a fundamental system of neighborhoods of c.

All singletons of $\mathbb{R} \times]0,1]$ are defined to be open.

The space X clearly is a Moore space. The sequence $(\mathcal{H}(n))_{n\in\mathbb{N}}$ of open covers

$$\mathcal{H}(n) = \{V_k(c) \mid x\in\mathbb{R}, c\in C_x \ k\geq n\} \cup \{\{(y,z)\} \mid y \in \mathbb{R},\ z \in]0,1]\}$$

is a nested development for the space.

It can easily be seen that the space X is locally completely metrizable. The basic properties of the example are stated in the following theorem.

Theorem.

The space X is Rudin-complete, but not Moore-complete. X is completely regular but not normal.

Proof.

The first assertion is trivial since the collections $(\mathcal{H}(n))_{n\in\mathbb{N}}$ form a Rudin-complete development for X.

We shall now prove that X is not Moore-complete. In view of a result of [AL2] stating that for any nested development $(G(n))_{n\in\mathbb{N}}$ of a Moore-complete space there are subcollections $G'(n)$ of $G(n)$, $n \in \mathbb{N}$ such that $(G'(n))_{n\in\mathbb{N}}$ is a Moore-complete development, it suffices to show that there are no subcollections $\mathcal{H}'(n) \subset \mathcal{H}(n)$ such that $(\mathcal{H}'(n))_{n\in\mathbb{N}}$ is a Moore-complete development. Suppose on the contrary that there are subcollections $\mathcal{H}'(n) \subset \mathcal{H}(n)$ such that $(\mathcal{H}'(n))_{n\in\mathbb{N}}$ is Moore-complete.

For $n \in \mathbb{N}$ let P_n be the following property for real numbers. The number x has property P_n if there exist a $H_n \in \mathcal{H}'(n)$ and a $z > 0$ such that $](x,0),(x,z)] \subset H_n$. Then in each bounded interval all but a finite number of points have the property P_n. This assertion follows from the maximality of the families C_x. So we can construct a descending chain of closed intervals $(V_n)_{n\in\mathbb{N}}$ such that all points of V_n satisfy P_n. Take a point $p \in \bigcap_{n\in\mathbb{N}} V_n$ and for $n \in \mathbb{N}$ consider the closed set $M_n =](p,0)(p,z_n)] \subset H_n$, where $(z_n)_n$ is a monotonic sequence converging to 0.

Moore completeness of X would imply that $\bigcap_{n\in\mathbb{N}} M_n \neq \phi$, which is a contradiction.

That X is completely regular follows from the fact that the sets $V_n(c)$ are clopen.

We shall show that X is not normal.

Consider the disjoint closed subsets $Y_1 = C_0$ and $Y_2 = \bigcup_{n\in\mathbb{N}} C_{\frac{1}{n}}$.

Let W_1 and W_2 be neighborhoods of Y_1 and Y_2. Define a function $f_0 : C_0 \to \mathbb{N}$ such that $V_{f_0(c)}(c) \subseteq W_1$.

The set $A_0 = \{0\} \cup \{y \mid y = c_n \text{ for some } c \in C_0 \text{ and } n \geq f_0(c)\}$ is a neighborhood of 0 in the usual topology on IR as can be shown as follows. Suppose on the contrary that no interval containing 0 is included in A_0. Then for $m \in \mathbb{N}$ choose $d_m \in]-\frac{1}{m},\frac{1}{m}[$ such that $d_m \notin A_0$. The sequence $d = (d_m)_{m\in\mathbb{N}}$ converges to 0 but $d \notin C_0$ and for any $c \in C_0$ we have that $d \cap c$ is finite. This fact contradicts the maximality of C_0. Analogously we define functions $f_{\frac{1}{n}} : C_{\frac{1}{n}} \to \mathbb{N}$ such that for $c \in C_{\frac{1}{n}}$ we have $V_{f_{\frac{1}{n}}(c)}(c) \subset W_2$ and show that the corresponding sets $A_{\frac{1}{n}}$ are neighborhoods of $\frac{1}{n}$ on IR. It follows that we can find an $m \in \mathbb{N}$ and a point $p \in (A_0 \setminus \{0\}) \cap (A_{\frac{1}{m}} \setminus \{\frac{1}{m}\})$. This implies that $W_1 \cap W_2 \neq \phi$.

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