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In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [344]--347.

Persistent URL: http://dml.cz/dmlcz/700603

COUNTABLE SMALL RANK AND CARDINAL INVARIANTS

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The concept of small rank was introduced by A. V. Arhangel'skii in [1] and [2] shortly after the closely related concept of rank was introduced by J. Nagata in [6]. In these papers it proved its worth in the study of dimension theory, especially of metric spaces, as did that of rank. In addition, spaces with various kinds of bases, all having countable rank, have contributed in a fundamental way to metrization theory [1], [3], [4], [5], [6].

In this paper I will show how closely related small rank (and similar, stronger properties) on a space are to certain fundamental cardinal invariants, such as spread, caliber, and height.

To begin with, I would like to propose a reform in the current terminology of partially ordered sets. The term "antichain" should naturally refer to a collection of pairwise <u>incomparable</u> members of a poset. Current jargon has it denoting a collection of pairwise incompatible members, for which I propose adopting the term "strong antichain". What this means, for collections of subsets of a set, is that in an antichain no member is contained in any other, while a strong antichain is a disjoint collection.

A collection of subsets of a set will be called <u>irreducible</u> if each member contains at least one point not in any of the others; obviously, this concept is intermediate between that of an antichain and that of a strong antichain. A collection will be called <u>fixed</u> if it has nonempty intersection.

1.1. <u>Definition</u>. A collection of subsets of a set X is <u>of countable small rank</u> if every fixed irreducible subcollection is countable; it is <u>of countable rank</u> if every fixed antichain is countable.

These concepts have obvious generalizations to higher cardinals, as will the results of this section.

By taking binary unions with a fixed member of B, one easily proves:

1.2. <u>Lemma</u>. Let B be a collection of subsets of a set X, closed under finite union. B is of countable small rank if, and only if, every irreducible subcollection is countable.

The analogous statement for countable rank does not hold.

The following theorem is a trivial consequence of Lemma 1.2.

1.3. Theorem. The following are equivalent for a space X :
(1) X has countable spread (that is, every discrete subspace of X is countable).
(2) Every irreducible collection of open subsets of X is countable.
(3) X has a base in which every irreducible collection is countable.
(4) The collection $\mathcal{T}$ of open subsets of X is of countable small rank.
(5) Every base for $\mathcal{T}$ is of countable small rank.

A space can easily have a base of countable small rank and not be of countable spread. For example, in any linearly ordered space, a fixed irreducible collection of open intervals can have at most two members! Also, any σ-disjoint base is of countable rank. But there is an interesting condition under which the relationship holds.

1.4. Theorem. Let X be a space of caliber $\aleph_1$. Then X has countable spread if, and only if, it has a base of countable small rank.

Proof. A space is of caliber $\aleph_1$ if, and only if, every point-countable collection of open sets is countable. If these sets are taken from a base of countable small rank, then any irreducible collection will be point-countable, hence countable. Now use 1.3 (3).

A separable space is of caliber $\aleph_1$, so:

1.5. Corollary. Every separable space with a base of countable small rank is of countable spread.

To put it another way: a separable space has a base of countable small rank if, and only if, every base is of countable small rank.

As countable spread is to bases of countable small rank, so the property of being hereditarily Lindelöf is to bases which are point-additively $\aleph_0$-Noetherian:

1.6. Definition. A collection B of subsets of a set X is $\aleph_0$-Noetherian if every ascending well-ordered sequence of members of B is countable. It is [point]-additively $\aleph_0$-Noetherian if for every [fixed] subset B' of B , there exists a countable collection $B'' \subset B'$ such that $\bigcup B''$ contains every member of B'.

In analogy with Lemma 1.2, we have (1) of the following:

1.7. Lemma. (1) Let B be a collection of subsets of a set X , closed under finite union. B is point-additively $\aleph_0$-Noetherian if, and only if, it is additively $\aleph_0$-Noetherian.
(2) If B is closed under countable union, it is $\aleph_0$-Noetherian if, and only if, it is [point]-additively $\aleph_0$-Noetherian.

Also, it is clear that every point-additively $\aleph_0$-Noetherian collection is of countable small rank. The analogy between the concepts continues with:

1.8. Theorem. The following are equivalent for a space X :

(1) X is hereditarily Lindelöf.

(1′) Every open subspace of X is Lindelöf.

(2) $\mathcal{T}$ is additively $\aleph_0$-Noetherian.

(3) $\mathcal{T}$ has an additively $\aleph_0$-Noetherian base.

(4) $\mathcal{T}$ is point-additively $\aleph_0$-Noetherian.

(5) Every base for $\mathcal{T}$ is point-additively $\aleph_0$-Noetherian.

(6) $\mathcal{T}$ is $\aleph_0$-Noetherian.

We even have analogues of Theorem 1.4 and Corollary 1.5. But there is no need to stop there:

1.9. Theorem. Let X be a space of countable spread. Then X is hereditarily Lindelöf if, and only if, it has a point-additively $\aleph_0$-Noetherian base.

Proof. Every space with a point-additively $\aleph_0$-Noetherian base is (hereditarily) meta-Lindelöf: if we take a basic open cover, well-order it, and remove the subsequence of all sets contained in the union of the preceeding ones, the result is a point-countable cover. Now it is well known that every meta-Lindelöf space of countable spread is Lindelöf.

There are no known examples, using ZFC alone, of a non-Lindelöf regular space of countable spread. In other words, there is no "real" example of a regular space in which every fixed irreducible collection of open sets is countable, without it being also true that for every [fixed] collection of open sets there is a countable subcollection covering the same points. But when it comes to bases, the story is quite different:

1.10. Example. The space ω_1 of countable ordinals is countably compact but not compact, and hence not meta-Lindelöf. Therefore, it does not have a point-additively $\aleph_0$-Noetherian base. On the other hand, being a linearly ordered space, it has a base of small rank two: every fixed, irreducible collection of intervals contains at most two members.

And yet, here is a simple result that makes point-additively $\aleph_0$-Noetherian sound almost like countable small rank:

1.11. Lemma. A collection $\mathcal{B}$ of subsets of a set X is [point-] additively $\aleph_0$-Noetherian if, and only if, every transfinite [fixed] well-ordered sequence of members of $\mathcal{B}$, each of which contains a point not in the preceding ones, is countable.

Finally, here are two easy results which bring out the distinction between $\aleph_0$-Noetherian and the other concepts.

1.12. Lemma. Let X be a space satisfying the countable chain condition (that is, every strong antichain of open sets is countable). If X has a base of regular open sets (for example, if X is regular) then this base is $\aleph_0$-Noetherian.

1.13. Corollary. Every regular space of countable spread has an $\aleph_0$-Noetherian base of countable small rank.

Bibliography

[1] A. V. Arhangel'skii: Ranks of systems of sets and dimensionality of spaces. Fund. Math. 52 (1963), 257-275. (Russian.)

[2] A. V. Arhangel'skii: K-dimensional metrizable spaces. Vestnik Moskov. Univ. Ser I Mat. Meh. no. 2 (1962), 3-6. (Russian.)

[3] A. V. Arhangel'skii and V. V. Filippov: Spaces with bases of finite rank, Math. USSR - Sb. 16 (1972), 147-158.

[4] G. Gruenhage and P. Nyikos: Spaces with bases of countable rank. General Topology and Appl. (to appear).

[5] G. Gruenhage and P. Zenor: Metrization of spaces of countable large basis dimension. Pacific J. Math. 59 (1975), 455-460.

[6] J. Nagata: On dimension and metrization. In: General Topology and its Relations to Modern Analysis and Algebra, New York, Academic Press, 1962.

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