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In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. 113--115.

Persistent URL: http://dml.cz/dmlcz/700609

ON THE GEOMETRIC CHARACTERIZATION OF DIFFERENTIABILITY

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The aim of this contribution, which is a brief survey of my papers [2,3,4], is to give three types of geometric characterization of differentiability of mappings in Banach spaces. (Till now, this problem was successfully solved in finitely dimensional spaces only.)

Throughout the paper, let Z be a Banach space, $S=\{z\in Z: \|z\|=1\}$ and $B_r^Z=\{z\in Z: \|z\|<r\}$; the graph of a mapping F will be denoted by G(F).

The first type of the characterization mentioned above is the characterization by means of a tangent cone to the graph of a mapping. To this aim, the notion of a tangent cone to a set will be introduced as follows:

<u>Definition 1.</u> Let (C_a, $a\in A$) be a net of cones in Z with a common vertex $z_0\in Z$. The conic limit of C_a is defined as the union of the one-point cone $\{z_0\}$ with all cones $C\subset Z$ with a vertex at z_0 and with the property: for every $\varepsilon>0$, there is $a_0\in A$ such that $C\subset U_\varepsilon(C_a)$ and $C_a\subset U_\varepsilon(C)$ whenever $a\succ a_0$, where

$$U_\varepsilon(C) = \{z : z=z_0+k(z'-z_0),\ k\geq 0,\ z'\in(C\cap(z_0+S))+B_\varepsilon^Z\}$$

is the conic ε-neighbourhood of C. We denote this limit by $\text{c-}\lim\limits_{a\in A} C_a$ and call it regular if it contains more than one point.

<u>Definition 2.</u> Let $M\subset Z$ be a non-empty set and $z_0\in\overline{M}$. Denoting

$$C_r(M,z_0) = \{z : z=z_0+k(z'-z_0),\ k\geq 0,\ z'\in M,\ \|z'-z_0\|<r\}$$

for $r>0$, the set

$$C_0(M,z_0) = \text{c-}\lim_{r\to 0} C_r(M,z_0)$$

is said to be the tangent cone to M at the point z_0.

The tangent cone defined in this way is always a non-empty closed cone with a vertex z_0 (it may degenerate to the one-point cone $\{z_0\}$ in the case of an irregular limit). It is in a close connection with similar notions of some other authors [1,5,6,8] but nevertheless, there is a difference there that makes possible to characterize Fréchet differentiability of mappings also in infinitely dimensional spaces.

<u>Theorem 1.</u> Let X,Y be Banach spaces, $D\subset X$, x_0 an interior point of D and let $F:D\to Y$ be a mapping. Then F possesses the Fréchet derivative $F'(x_0)$ at x_0 if and only if F is continuous at x_0 and there is a continuous linear mapping $L:X\to Y$ so that

$$C_o(G(F),\ (x_o,F(x_o))) = (x_o,F(x_o)) + G(L)\ ;$$

in this case, it is $F'(x_o)=L$.

See [4] for the proof and other details.

The second method of the geometric characterization of differentiability is based on the notion of a tangent. Our concept is a generalization of the finitely dimensional one given in [7].

Definition 3. Let C be a cone in Z with a vertex at $z_o \in Z$, let H be a linear manifold in Z of co-dimension 1 such that $z_o \in H$. The number $d = \operatorname{dist}(C \cap (z_o+S), H)$ is called the deviation of C from H, the set $C' = Z \setminus (C \cup (-C))$ is called the co-cone to C (in Z) and, denoting by $\mathbb{C}_{H,d}(z_o)$ the system of all cones in Z with a vertex at z_o and with a deviation d from H, the set

$$C^{\prime}_{H,d}(z_o) = \bigcap \{ \overline{C}' : C' \text{ is a co-cone to some } C \in \mathbb{C}_{H,d}(z_o) \}$$

is called the circular co-cone in Z with a vertex at z_o and a co-deviation d from H (it is a co-cone to some cone in Z, too).

It is easy to see that

$$C^{\prime}_{H,d}(z_o) = \{ z : z=z_o+k(z'-z_o),\ k \geq 0,\ \|z'-z_o\|=1,\ \operatorname{dist}(z',H) \leq d \}.$$

Definition 4. Let X,Y be Banach spaces, $D \subset X$, $F:D \to Y$, x_o an interior point of D and let P be a closed linear manifold in Z. The manifold P is said to be tangent to the graph G(F) of F at the point $z_o=(x_o,F(x_o))$ iff F is continuous at x_o, $P-z_o$ is a graph of some continuous linear mapping from X into Y and for every $d>0$, there is $r(d)>0$ such that

$$G(F) \cap (z_o + B^{X\times Y}_{r(d)}) \subset \bigcap_{H \in \mathbb{H}} C^{\prime}_{H,d}(z_o)\ ,$$

where $\mathbb{H}$ is the system of all closed linear manifolds H in XxY of co-dimension 1 having the property $P \subset H$.

Now, the following theorem holds:

Theorem 2. Let X,Y be Banach spaces, $D \subset X$, $F:D \to Y$ and let x_o be an interior point of D. The mapping F possesses the Fréchet derivative at the point x_o if and only if there exists a tangent manifold to the graph of F at the point $(x_o,F(x_o))$.

See [3] and [2] for the proof and other details.

Finally, we comes to the third characterization of differentiability. In fact, it is a formal extraction of a basic approximation idea from the preceding characterization (see [2] for details).

Definition 5. Let P be a linear manifold in Z, $z_o \in P$ and $\varepsilon > 0$. The set

$$C(P,z_o,\varepsilon) = \{ z : \operatorname{dist}(z,P) \leq \varepsilon \|z-z_o\| \}$$

is called the ε-cone of P in Z with a vertex z_o.

Theorem 3. Let X,Y be Banach spaces, $D \subset X$, $F: D \to Y$, x_0 an interior point of D. Then F is Fréchet differentiable at x_0 if and only if F is continuous at x_0 and there is continuous linear mapping $L: X \to Y$ such that for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$G(F) \cap (z_0 + B_\delta^{X \times Y}) \subset C(G(L), z_0, \varepsilon) .$$

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