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TOPOLOGICAL CONSIDERATIONS IN THE FOUNDATIONS
OF QUANTUM AND TIME THEORIES

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Abstract I. P. Jordan, J. v. Neumann and E. Wigner have shown the power of an algebraic approach to quantum theories [6] . The greater weakness of the above work was that the set of observables has a finite linear basis. This set was later equipped with a convenient topological structure. The first attempt was done by von Neumann [10] . From a mathematical point of view the topology induced by von Neumann was lacking a plausible phenomenological motivation. In fact, the concept of state played no a substantial role in that work. Since the concept of state is of great importance for the physical motivation, we equipe the set $\mathcal{O}$ of observables with the natural topology.

II. In the second part of this paper we define in a simpler way the chronological topology $\mathcal{T}_{\text{chron}}$ and then we give some criteria and conditions so that $\mathcal{T}_{\text{man}} = \mathcal{T}_{\text{chron}}$. Some different concepts of time are then presented and criticized.

Part I. We consider the following well known axioms and definitions :

Axiom I. We can correspond to each physical system a triple $(\mathcal{O}, \mathcal{Y}, (|))$, which consists of the set $\mathcal{O}$ of all of its observables, the set $\mathcal{Y}$ of all of its states and $(\;|\;) : \mathcal{O}\times\mathcal{Y} \to \mathbb{R}$, which corresponds to each pair $(O,\psi)\in\mathcal{O}\times\mathcal{Y}$ a real number $(\psi|O)$ interpreted as the expectation value of the observables O, when the system is at the state ψ.

Definition 1. A subset $\mathfrak{X}\subseteq\mathcal{Y}$ is called full with respect to a subset $\mathcal{J}\subset\mathcal{O}$ iff

$$(\forall O_1\in\mathcal{J})(\forall O_2\in\mathcal{J})\left[\left(O_1/_{\mathfrak{X}} \leqslant O_2/_{\mathfrak{X}} \Leftrightarrow (\forall\psi\in\mathcal{Y})\left[(\psi/O_1)\leqslant(\psi/O_2)\right]\right) \Rightarrow O_1 \leq O_2\right],$$

where $O/_{\mathfrak{X}}$ is the restriction of the map $(\psi|O) : \mathcal{Y}\to\mathbb{R}$ to the subset $\mathfrak{X}\subseteq\mathcal{Y}$.

Axiom II. The above defined binary relation is a partial order relation. In particular, it holds $(O_1 \leqslant O_2 \wedge O_1 \geqslant O_2) \Rightarrow O_1 = O_2$.
If $O_1, O_2\in\mathcal{O}$, then the family $\{(\psi|O_1)+(\psi|O_2)\}_{\psi\in\mathcal{Y}}$ defines an observable . [1] , [2] , [10] .

Axiom III. (i). $(\exists\, \mathbb{O}\in\mathcal{O})(\exists\, \mathbb{1}\in\mathcal{O})(\forall\psi\in\mathcal{Y})\left[(\psi|\mathbb{O}) = 0 \wedge (\psi|\mathbb{1}) = 1\right]$.
(ii). $(\forall O\in\mathcal{O})(\forall k\in\mathbb{R})(\exists\, kO\in\mathcal{O})(\forall\psi\in\mathcal{Y})\left[(\psi|kO) = k(\psi|O)\right]$.
(iii). $(\forall (O_1,O_2)\in\mathcal{O}\times\mathcal{O})(\exists\, O_1+O_2\in\mathcal{O})(\forall\psi\in\mathcal{Y})\left[(\psi|O_1+O_2) = (\psi/O_1)+(\psi|O_2)\right]$. [1] , [2].

This axiom provides the $\mathcal{O}$ with a real vector space structure. On the basis of definitions of the addition and multiplication by scalars the elements of $\mathcal{Y}$ are real linear functionals on $\mathcal{O}$.

Definitions 2. Let $\mathcal{Y}_0$ be the set of all dispersion free states on $\mathcal{O}$. A subset $\mathfrak{X} \subseteq \mathcal{Y}$ is called complete iff it is full with respect to the subset

$$\mathcal{O}_{\mathfrak{X}} \equiv \{ O \in \mathcal{O} : \mathcal{Y}_0 \supseteq \mathfrak{X} \} \subseteq \mathcal{O},$$

deterministic for a subset $\mathcal{J} \subseteq \mathcal{O}$ iff $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$ and compatible iff the set

$\mathcal{Y}_{\mathcal{J}} = \bigcap_{\beta \in \mathcal{J}} \mathcal{Y}_{\beta}$ is complete.

We can correspond to each $(O_1, O_2) \in \mathcal{O} \times \mathcal{O}$ an element $O_1 * O_2 \in \mathcal{O}$, called the symmetrized product of O_1 and O_2, which is defined, as follows:

$$O_1 * O_2 = \tfrac{1}{2} \left[(O_1 + O_2)^2 - O_1^2 - O_2^2 \right].$$

We define, for each triple $(O_1, O_2, O_3) \in \mathcal{O}^3$, the associator, as follows:

$$\{ O_1, O_2, O_3 \} = (O_1 * O_2) * O_3 - O_1 * (O_2 * O_3).$$

If $\mathcal{J}$ is any subset of $\mathcal{O}$, then $\mathcal{Y}_{\mathcal{J}} = \bigcap_{\beta \in \mathcal{J}} \mathcal{Y}_{\beta}$ and $\mathcal{O}_{\mathcal{Y}_{\mathcal{J}}} \equiv \mathcal{O}(\mathcal{J})$.

Axiom IV. For each $(O_1, O_2, O_3) \in \mathcal{O}^3$, in which O_1, O_3 are compatible, the associator $\{O_1, O_2, O_3\}$ vanishes.

The set $\mathcal{O}$ is an abelian Jordan algebra. [2] .

Definitions 3. For every $O \in \mathcal{O}$ and every complete set $\mathfrak{X} \subseteq \mathcal{Y}_0$, we define

(i). $w(O, \mathfrak{X}) \equiv \{ \alpha \in \mathbb{R}_+ : (\forall \psi \in \mathfrak{X}) [|(\psi | O)| < \alpha] \} \subset \mathbb{R}_+$ and

(ii). $N_{\mathfrak{X}}(O) \equiv g.l.b. \{ w(O, \mathfrak{X}) \} \in \mathbb{R}_+$.

Theorem 1. $N_{\mathfrak{X}}(O)$ is independent of $\mathfrak{X} \subseteq \mathcal{Y}_0$ and it is equal to $N(O) \equiv \sup_{\psi \in \mathcal{Y}} |(\psi | O)|$.

Remark 1. From the above theorem follows that

$$(\forall k \in \mathbb{R}) (\forall (O_1, O_2) \in \mathcal{O} \times \mathcal{O}) \left[N(kO_1) = |k| \cdot N(O_1) \wedge N(O_1 + O_2) \leq N(O_1) + N(O_2) \right]$$

and the vanishing of $N(O)$ occurs only whenever $O = \mathbf{0}$.

Hence holds the following corollary:

Corollary N is a norm on $\mathcal{O}$ and

$$(\forall (O, \psi) \in \mathcal{O} \times \mathcal{Y}) \left[|(\psi | O)| \leq N(O) \right].$$

Remark 2. $\mathcal{O}$ is now equipped with a real Banach space structure relative to the natural norm introduced above, and the states $\psi \in \mathcal{Y}$ are positive linear continuous functionals on $\mathcal{O}$ with respect to the topology induced by this norm. We shall now extend $\mathcal{Y}$ to the set of all such functionals on $\mathcal{O}$.

Axiom V. The norm of any element $O \in \mathcal{O}$ is finite and $\mathcal{O}$ is complete, if it is considered as a metric space with distance between two any elements $O_1, O_2 \in \mathcal{O}$, defined by $N(O_1 - O_2)$. Hence $\mathcal{Y}$ is identified with the set of all continuous positive linear functionals ψ on $\mathcal{O}$, which satisfy the condition $(\psi | \mathbf{I}) = 1$.

Main theorem. The following statements hold:

$$(\forall \psi \in \mathcal{Y}) \ (\forall O_1 \in \mathcal{O}) \ (\forall O_2 \in \mathcal{O})$$

(i). $\left[|(\psi | O_1 * O_2)| \leq N(O_1) \cdot N(O_2) \right]$.

(ii). $[N(O_1 * O_2) \le N(O_1) \cdot N(O_2)]$.

(iii). $[N(O_1^2 - O_2^2) \le N(O_1 - O_2) \cdot N(O_1 + O_2)]$.

In order to prove the main theorem, we use the following lemmas :

Lemma 1. $(\forall O_1 \in \mathcal{O})(\forall O_2 \in \mathcal{O})[N(O_1^2) = N^2(O_1) \wedge N(O_1^2 - O_2^2) \le \max\{N^2(O_1), N^2(O_2)\}]$.

Lemma 2. (i). The Schwarz inequality holds :

$$(\forall \psi \in \mathcal{Y})(\forall (O_1, O_2) \in \mathcal{O}^2)[(\psi | O_1 * O_2) \le (\psi | O_1^2) \cdot (\psi | O_2^2)].$$

(ii). Let O_1 be any element of $\mathcal{O}$ and $\psi \in \mathcal{Y}_{O_1}$. Then

$$(\forall O_2 \in \mathcal{O})[(\psi | O_1 * O_2) = (\psi | O_1) \cdot (\psi | O_2)].$$

Proof of main theorem. From Schwarz inequality and the previous results we have

$$(\psi | O_1 * O_2)^2 \le (\psi | O_1)^2 (\psi | O_2)^2 \le N(O_1^2) \cdot N(O_2^2) = N^2(O_1) \cdot N^2(O_2).$$

This result proves the statement (i)

The statement (ii) follows directly from (i).

From the distributivity of the symmetrized product follows

$$O_1^2 - O_2^2 = (O_1 - O_2) * (O_1 + O_2).$$

From (ii) and the above property follows the truth of (iii).

Remarks 3. a. The statement (ii) expresses the continuity of the symmetrized product with respect to each of its factors.

b. The statement (iii) expresses the continuity of O_1^2 with respect to O_1 .

Part II. Topology and time theories.

Definition. A space - time M is a real four - dimensional connected C^∞ Hausdorff manifold with a globally defined C^∞ tensor field g of type (0,2), which is non-degenerate and Lorentzian. By Lorentzian (or hyperbolic normal) is meant that for any $x \in M$ there is a basis in $T_x = T_x(M)$ (the tangent space of M at x) relative to which g_x has the matrix diag (1, -1, -1, -1). [5] .

The metric would be used only for the definition of the structure of the null - cone of M, namely:

Let M be a space - time and $x \in M$. Any tangent vector $\upsilon \in T_x$ is said to be: timelike, spacelike or null according as $g_x(\upsilon, \upsilon)$ $(= g_{x\alpha\beta} X^\alpha X^\beta)$ is positive, negative or zero respectively. The null cone K_x at x is the set of null vectors in T_x . The above definition means that: a tangent vector $\upsilon \in T_x$ is timelike (null, spacelike) iff υ lies in the interior (respectively on, exterior) of K_x . The cone K_x disconnects the timelike vectors into two separate components. A space - time is called time - orientable (or temporally - orientable) if it is possible to make a consistent continuous choice all over M, of one component of the set of timelike vectors at each point of M. To label the timelike vectors so chosen future - pointing and the remaining ones past - pointing is to make the space - time M time - oriented. A space-time is clearly time - orientable if there exists a nowhere vanishing timelike vector field. The converse is also true. This follows from general theorems on the existence of cross - sections of fiber bundles (the fiber consisting of future - pointing timelike vectors at a point being "solid") [8] .

Once a choice of orientation has been made on M, then an orientation is induced on certain curves on M as follows:

A smooth parametrized curve γ of M is a future directed timelike (causal) curve iff the positive tangent vector to γ at every point x lies in the future lobe of K_x (lies inside or on the future lobe of K_x).

Definition of a causal space. A causal space is a quadruple $(M, \prec, \precsim, \nearrow)$, where M is a point set (of course with cardinality at least $\aleph_1$), and $\prec, \precsim, \nearrow$ are three relations on M, which, for all $x, y, z \in M$, satisfy the following axioms:

I. $x \precsim x$

II. $(x \precsim y \wedge y \precsim z) \Rightarrow x \precsim z$

III. $(x \precsim y \wedge y \precsim x) \Rightarrow x = y$

IV. $x \not\prec x$

V. $x \precsim y \Rightarrow x \prec y$

VI. a. $(x \precsim y \wedge y \prec z) \Rightarrow x \prec z$

b. $(x \prec y \wedge y \precsim z) \Rightarrow x \prec z$

VII. $x \nearrow y \Leftrightarrow (x \precsim y \wedge x \not\prec y)$ [7]

Examples. Minkowski space, Lanczos universe, de Sitter universe.

Counterexample. Einstein universe

We introduce the following definitions:

$I_r(x) \equiv \{y \in M : x \prec y\}$: the chronological future of x .

$I_\ell(x) \equiv \{y \in M : y \prec x\}$: the chronological past of x .

$J_r(x) = \{y \in M : x \precsim y\}$: the causal future of x .

$J_\ell(x) = \{y \in M : y \precsim x\}$: the causal past of x .

It is possible to introduce the following relations $\precsim$ and $\nearrow$, as follows:

a. $x \precsim y \Leftrightarrow (I_r(x) \supset I_r(y) \wedge I_\ell(x) \subset I_\ell(y))$.

b. $x \nearrow y \Leftrightarrow (x \precsim y \wedge x \not\prec y)$.

$x \precsim y$ and $x \prec y$, $x, y \in M$, mean respectively, that x causally preceeds y, and x chronologically preceeds y , i.e. $x \precsim y$ iff there is a future directed affine causal curve of nonzero length from x to y , and $x \prec y$ iff there is a future directed affine curve of nonzero length from x to y. Now $B(x) \equiv J_\ell(x) \cup J_r(x)$ is the set of points causally connectible with x in the sense that it is physically possible for a causal signal of nonnull length to connect y with x in case $y \in B(x)$ The events are considered as unextended in the classical sense; if space - time is taken as primitive, they are events which are localized at a space - time point.

Theorem. Let $(M, \prec)$ a prerdered set, which satisfies the following property:

$$(\forall (x,y) \in M \times M)\left[I_r(x) = I_r(y) \wedge I_\ell(x) = I_\ell(y) \Rightarrow x = y \right].$$

The quadruple $(M, \prec, \precsim, \uparrow)$, where $\precsim$, $\uparrow$ are the above introduced relations, is a causal space.

Remark 1. The preorder $\precsim$ guarantees us that twisted light cones are excluded, because $x \prec y$ and $y \prec x$ would imply the contradiction that $x \prec x$.

We denote by capital letters X, Y, Z events, which take place at the space - tine points x, y, z, respectively. An obvious and effective criterion of spatio - temporal coincidence,which can be formulated in terms of causal connectibility, is the following:

(C_1) : X and Y are spatiotemporally coincident iff, for every Y, Y is causally connectible with X, iff Y is causally connectible with Y. [4] , [7] .

(C_1) is sufficient only for those of the space - times, which sarisfy the following condition:

(C_2) : $\quad (\forall (x,y) \in M \times M)\left[J_\ell(x) = J_\ell(y) \wedge J_r(x) = J_\ell(y) \Rightarrow x = y \right]$, [9] .

Condition (C_2) is not satisfied in many space - times, encountered in the general relativity. Any space - time in which (C_2) fails is physically unreasonable. A space-time which contains closed causal curves violates our everyday conception of causation. But M need not contain closed causal curves in order for (C_2) fail. It is true however, that if (C_2) fails, the following condition will not hold for every $x \in M$ (it is equivalent to say that M will contain "almost closed" causal curves).

(C_3) : Every open neighborhood in $\mathcal{T}_{man}$ of x contains an open neighborhood which is not entered twice by any causal curve.

The chronological topology $\mathcal{T}_{chron}$ of space - time M is the coarset topology, in which $I_\ell(x) \cap I_r(y)$ are open, where x, y range M.

If M satisfies a condition somewhat weaker than (C_3) that is

(C_4) : M contains no almost closed timelike curves, then $\mathcal{T}_{chron} = \mathcal{T}_{man}$. Conversely, if $\mathcal{T}_{chron} = \mathcal{T}_{man}$, then (C_4) holds [7] .

Some different concepts of time. A. According to the Ancient Greeks' picture of the world, space - time M was a Cartesian product of time T and space S [11] : to any event one could assocciate an instant of time t and a location in space s; both time and space were absolute.

B. In Newtonian physics, space - time M may be represented as a product T × S in many ways. Space is relative because there is no absolute method of ascertaining whether or not two non - simultaneous events happen at the same place. In other words, there is no natural horizontal slicing of M; there is only a vertical fibring corresponding to the projection $\pi : M \to T$, which associates to any event $x \in M$ the corresponding instant of time $t = \pi(x)$; or time is absolute.

C. Time in the Galilean space. We denote by $\mathcal{G}al$ the Galilean category, which has Galilean spaces as objects. A Galilean space is an affine space $(M, V, +)$

endowded with a bilinear map $\varphi: V^* \times V^* \to \mathbb{R}$, which is (i) symmetric, (ii) positive, and (iii) of rank $n-1$, where $n = \dim V$. M is the underlying set, V is the associated vector space and + denotes the transitive and free action of the additive group V in M. If $\mathcal{A}$ and $\mathcal{B}$ are two categories, we denote by τ a covariant functor. If $(M_1, V_1, +, \varphi_1)$ and $(M_2, V_2, +, \varphi_2)$ are two Galilean spaces, the affine morphism f is a Galilean morphism if $\varphi_1 \circ (\tau f)^* = \varphi_2$. A Galilean automorphism is called a Galilean transformation. Let $A \subset V^*$ be the null space of φ and $B \subset V$ the subspace of all vectors orthogonal to A. For any Galilean space $G = (M, V, +, \varphi)$ the quotient space $T = M/S$ is called absolute time of G. If $\pi: M \to T$ is the canonical projection, then (M, T, π) is a fiber bundle with B as the typical fiber. The relation of absolute time to Galilean transformations is described by the following theorem:

Theorem. Let $\mathcal{B}un$ be the category of differentiable bundles. There is a covariant functor $\sigma: \mathcal{G}al \to \mathcal{B}un$ defined by $\sigma(G) = (M, T, \pi)$ and $\sigma(f) = (f, \alpha)$, where $f: M_1 \to M_2$ is a Galilean morphism and $\alpha: T_1 \to T_2$ is the unique map satisfying $\pi_2 \circ f = \alpha \circ \pi_1$.

D. The concepts of open and closed time. We assume that a space - time M, which is considered time - orientable, can be partitioned by a family of spacelike hypersurfaces [3] . This allows one to namufacture a time T: T is the quotient of M by the equivalence relation $x\mathcal{R}y$ which holds between $x, y \in M$ iff x and y lie on the same hypersurface, i.e. an instant of $\mathbf{T}$ is an $\mathcal{R}xy$ - equivalence class of points of M. Now if no future (or past) directed timelike curve of M intersects more than anyone of the hypersurfaces corresponding to the instants of T, we can define a mapping $F: T \to \mathbb{R}$ such that:

(i). The fibers of F are in $1-1$ correspondence with the instants of T and

(ii). for any $x, y \in M$, if there is a future directed timelike curve of nonzero length from x to y, then $F(x) < F(y)$.

In this case, T is called open.

Temporal separation. It is intuitively clear that for the model of tetradic relation " X, Y separate Z, W ", on a circle K means that: to get from Z to W one has to go around the circle K, passing over X or over Y.

Analogously, we say that T is closed if there is a mapping $G: M \to K$ (K = circle) such that:

(i) the fibers of G are in $1-1$ correspondence with the instants of T. and

(ii) for any $w, x, y, z \in M$, if these points are distinct and there is a future directed timelike curve from w to x, from x to y, and from y to z, then either this curve intersects all of the hypersurfaces of M which correspond to the instants of T or else the points $G(w)$, $G(y)$ separate the points $G(x)$, $G(y)$ on the circle K.

Remark 2. Some interesting solutions of the field equations of general relativity cannot be partitioned by spacelike hypersurfaces [3] and other solutions cannot accept a unique global space - time hypersurfaces or "time slice". In these cases time T cannot even be defined, and problems of time openess and / or time closedness do not arise. There are examples of relativistic space - times, from which we can manufacture a time T, which satisfies neither the above openess nor closedness criterion.

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