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# ON A THEOREM OF MICHAEL - MORITA - HANAI

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Morita and Hanai [13] proved that every closed continuous mapping from a normal $T_1$- space onto a first-countable space is peripherally compact, that is, boundaries of point inverses are countably compact. Combining this result with a standard technique due to Vainstein [18], Morita and Hanai proved that every closed continuous image of a metrizable space is metrizable if it is first-countable. This was also proved by Stone [17]. The above result of Morita and Hanai was later improved by Michael [9] who proved that the result holds more generally for q- spaces instead of first-countable spaces. It is the purpose of the present note to further improve upon the result of Michael and to show that several other known results are improved as a consequence. The ideas and results developed will then be used to obtain an interesting connection between the locally finite sum theorem and the closure-preserving sum theorem. We say that the locally finite sum theorem holds for a property $\mathcal{P}$ if whenever $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed covering of X such that each $F_\alpha$ has $\mathcal{P}$, then X has $\mathcal{P}$. If in the above statement, locally finite be replaced by hereditarily closure-preserving, one gets what is known as the closure-preserving sum theorem. Hereditarily closure-preserving families were first used by Lasnev [8] when he gave the first internal characterisation of closed continuous images of metrizable spaces. $\{F_\alpha : \alpha \in \Lambda\}$ is said to be hereditarily closure-preserving if every family $\{A_\alpha : \alpha \in \Lambda\}$ with $A_\alpha \subseteq F_\alpha$ for each $\alpha \in \Lambda$, is closure-preserving. We made a detailed study of the locally finite sum theorem in [1, 14]. In [15], we studied the closure-preserving sum theorem and its several interesting consequences. Obviously, if the closure-preserving sum theorem holds for a property, then the locally finite sum theorem also holds for it. We will show that for A-spaces of Michael, the converse holds for closed hereditary properties.

All spaces are assumed to be $T_1$. The abbreviation CH will be used for closed hereditary properties and Bd A will mean the boundary of A.

## 1. The Michael-Morita-Hanai theorem

We shall first obtain some general theorems about some classes of closed continuous images of spaces and show that several known results can be improved as an application of our general results.

**Theorem 1.1.** Let $\mathcal{P}$ be a property such that $\mathcal{P}$ is (i) CH (ii) preserved under quasi-perfect maps. If $f:X \to Y$ is a continuous mapping from $X$ onto $Y$ such that $\mathrm{Bd}\, f^{-1}(y)$ is countably compact for each $y \in Y$, then $Y$ has $\mathcal{P}$ if $X$ has $\mathcal{P}$.

*Proof:* For each $y \in Y$, let $x_y \in f^{-1}(y)$ be arbitrarily fixed and let

$$G_y = \begin{cases} [f^{-1}(y)]^{o} & \text{if } \mathrm{Bd}\, f^{-1}(y) \neq \phi . \\ f^{-1}(y) \sim \{x_y\} & \text{if } \mathrm{Bd}\, f^{-1}(y) = \phi . \end{cases}$$

Let $X^* = X \sim \bigcup\{G_y : y \in Y\}$ and let $h:X^* \to X$ be the identity map. If $g = f \circ h$, then $g$ is a closed continuous mapping of $X^*$ onto $Y$ such that

$$g^{-1}(y) = \begin{cases} \mathrm{Bd}\, f^{-1}(y) & \text{if } \mathrm{Bd}\, f^{-1}(y) \neq \phi . \\ x_y & \text{if } \mathrm{Bd}\, f^{-1}(y) = \phi . \end{cases}$$

Thus $g$ is a quasi-perfect map from the closed subset $X^*$ of $X$ onto $Y$ and hence $Y$ has $\mathcal{P}$ in view of (i) and (ii).

Obviously, the proof of the above theorem shows that if $\mathrm{Bd}\, f^{-1}(y)$ is compact for each $y \in Y$, then the map $g$ is a perfect map and if $\mathrm{Bd}\, f^{-1}(y)$ is finite for each $y \in Y$, then $g$ is a finite-to-one closed continuous map. Hence we can have the following two theorems.

**Theorem 1.2.** If $\mathcal{P}$ is (i) CH (ii) preserved under perfect maps and if $f:X \to Y$ is a continuous mapping from $X$ onto $Y$ such that $\mathrm{Bd}\, f^{-1}(y)$ is compact for each $y \in Y$, then $Y$ has $\mathcal{P}$ if $X$ has $\mathcal{P}$.

**Theorem 1.3.** If $\mathcal{P}$ is (i) CH (ii) preserved under finite-to-one, closed continuous maps and if $f:X \to Y$ is a continuous map from

X onto Y such that Bd $f^{-1}(y)$ is finite for each $y \in X$, then Y has $\mathcal{P}$ if X has $\mathcal{P}$.

1.4. The following properties, which are at least closed hereditary, are preserved under quasi-perfect maps and hence Theorem 1.1 holds for all these:

Metrizability, the property of being a $\sigma$-space, the property of being a $\Sigma$-space, Čech-completeness, paracompactness, $\mathfrak{m}$-paracompactness, the property of being a normal M-space, the property of being a wM-space, the property of being an $M^*$-space, expandability, discrete H.C. expandability, the property of being a strongly paracompact Hausdorff locally Lindělof space.

1.5. The following properties, which are at least closed hereditary, are preserved under perfect maps and hence Theorem 1.2 holds for all these:

Regularity, local compactness, the property of being an $\aleph$-space, the property of being a space of countable type, countable compactness at infinity, paracompactness at infinity, countable paracompactness at infinity, normality at infinity.

1.6. The following properties, which are at least CH, are preserved under finite-to-one closed continuous maps and hence Theorem 1.3 holds for all these:

The property of being a q-space, an r-space, a bi-sequential or a first-countable space.

1.7. One can in fact talk of the following more general form of Theorems 1.1 and 1.2:

"If $f: X \to Y$ is a closed continuous mapping with Bd $f^{-1}(y)$ $\mathfrak{m}$-compact for each $y \in Y$ and if f is (i) CH (ii) preserved under closed continuous maps with fibres compact, then Y has $\mathcal{P}$ if X has $\mathcal{P}$." This general form of the theorem will apply to normal +M($\mathfrak{m}$)-spaces of Ishii [6] .

And now we require some definitions before we can state our next theorem which is the main result of the section.

1.8. A space X is called an A-space if for every decreasing sequence $\langle A_n \rangle$ of subsets of X such that $y \in \overline{A_n \sim \{y\}}$ for each n for some $y \in X$, there exist $B_n \subseteq A_n$ such that $\cup \overline{B}_n$ is not closed. This concept of A-spaces is due to Michael [11] and offers a generalisation of several known generalized forms of first countable spaces. For instance, countably biquasi-k spaces and hence spaces of pointwise countable type, q-spaces, locally compact spaces, locally countably compact spaces are all A-spaces.

1.9. A space X is said to have the ss-discrete property if for every countable, discrete closed subset $\{x_n : n = 1,\dots\}$ of X and for every collection $\langle U_n \rangle$ of open subsets of X such that $x_n \in U_n$, there is subsequence $\{x_{n_i} : i = 1,2,\dots\}$ of $\{x_n : n = 1,2,\dots\}$ and a locally finite collection $\{V_{n_i} : i = 1,2,\dots\}$ of open subsets of X such that $x_{n_i} \in V_{n_i} \subseteq U_{n_i}$ for each i. We shall refer to spaces with ss-discrete property as ss-discrete spaces. This notion of ss-discrete spaces is due to Isiwata and is mentioned in a paper by Morita [12]. Actually, for completely regular spaces, ss-discrete spaces are identical will well-separated spaces of Morita [12]. A space is well-separated in the sense of Morita if every infinite discrete closed set is not relatively pseudo-compact. This is equivalent to saying that the closure of every relatively pseudo-compact set is countably compact. Among the spaces that are ss-discrete are countably paracompact, discrete HC-expandable spaces, normal spaces, weakly normal spaces, topologically complete spaces, real-compact spaces, P-spaces and well-separated spaces in the sense of Hansard.

<u>Theorem 1.10.</u> If $f: X \to Y$ is a closed continuous mapping from an ss-discrete space X onto an A-space Y, then Bd $f^{-1}(y)$ is countably compact for each $y \in Y$.

<u>Proof:</u> Suppose there is a $y \in Y$ such that Bd $f^{-1}(y)$ is not countably compact. Then there is a discrete closed set $\{x_n : n = 1,\dots\}$ such that $x_n \in$ Bd $f^{-1}(y)$ for each n. Let $G_n = X \sim \{x_i : i \neq n\}$. Then $\{G_n : n = 1,2,\dots\}$ is a collection of open subsets of X such that $x_n \in G_n$ for each n. Since X has the ss-discrete property, there is a subsequence $\{x_{n_i} = y_i : i = 1,2,\dots\}$ and a locally finite collection $\{V_{n_i} = U_i : i = 1,2,\dots\}$ of open subsets of X such that $y_n \in U_n$ for each n. Now for each n, let $W_n = \bigcup_{i \geq n} \overline{U}_i$. Let $F_n = f(W_n)$. Now $\{\overline{U}_i : i = 1,2,\dots\}$ being locally finite, $W_n$ is a closed set. Thus $F_n$ is a closed set. Also, $\{W_n : n = 1,2,\dots\}$ is locally finite, because for each $x \in X$, x belongs to at most finitely many $U_1,\dots, U_n$ (say) and then $X \sim W_{n+1}$ is an open set containing x which intersects at most $W_1,\dots,W_n$. Also, $W_n$ is a neighbourhood of $y_n$,

since $y_n \in U_n \subseteq W_n$. Hence $y \in \overline{F_n \sim \{y\}}$ for each n. So $\{F_n : n = 1,2,\ldots\}$ is a decreasing sequence with a common accumulation point and hence there exist $B_n \subseteq F_n$ such that $\cup \overline{B}_n$ is not closed. Let $A_n = f^{-1}(\overline{B}_n) \cap W_n$. Then $\{A_n : n = 1,2,\ldots\}$ is a locally finite family of closed sets and hence $A = \bigcup_{n=1}^{\infty} A_n$ is closed. But now $f(A_n) \subseteq \overline{B}_n$ and $\overline{B}_n \subseteq f(A_n)$. Thus $\cup \overline{B}_n = f(A)$ which is closed. But this is a contradiction. Therefore Bd $f^{-1}(y)$ must be countably compact.

1.11. Michael [10] has shown that Bd $f^{-1}(y)$ is relatively pseudocompact without any assumption on X . This means that if we are dealing with completely regular spaces, then Bd $f^{-1}(y)$ would be countably compact if X is well-separated in the sense of Morita. And we have already remarked that in a completely regular space, ss-discrete spaces are equivalent to well separated spaces in the sense of Morita.

In view of Theorem 1,1, the result of the above Theorem 1.10 can now be used to conclude the following

<u>Theorem 1.12.</u> If $\mathcal{P}$ is (i) CH (ii) preserved under quasi-perfect maps and if $f: X \to Y$ is a closed continuous mapping of an ss-discrete space X onto an A-space Y, then Y has $\mathcal{P}$ if X has $\mathcal{P}$.

1.13. A space X is said to be isocompact if every closed, countably compact subset of X is compact. This notion is due to Bacon[2]. Some well known spaces which are isocompact are semi-stratifiable spaces and hence Moore spaces, developable spaces, semi-metric spaces and stratifiable spaces. Also, regular $\sigma$-spaces, Lindelöf spaces, meta Lindelöf spaces with point countable bases, pointwise paracompact and hence paracompact spaces, subparacompact spaces, screenable spaces are all isocompact. Topologically complete spaces, P-spaces and regular almost real-compact spaces are also isocompact.

<u>Theorem 1.14.</u> If $f: X \to Y$ is a closed continuous mapping from an ss-discrete and isocompact space X onto an A-space Y, then Bd $f^{-1}(y)$ is compact for each $y \in Y$.

<u>Proof:</u> Obvious, in view of Theorem 1.10 and the definition of an isocompact space.

Again, as before, combining Theorem 1,2 with Theorem 1.14 above, we have the following:

<u>Theorem 1.15.</u> If $\mathcal{P}$ is (i) CH (ii) preserved under perfect maps and if $F:X \to Y$ is a closed continuous mapping from an ss-discrete and isocompact space X onto an A-space Y, then Y has $\mathcal{P}$ if X has $\mathcal{P}$.

Since every P-space is ss-discrete and also isocompact and every compact set is finite in a P-space [5] , we can also state the following:

<u>Theorem 1.16.</u> If $\mathcal{P}$ is (i) CH (ii) preserved under finite-to-one, closed continuous maps and if $f:X \to Y$ is a closed continuous mapping from a P-space X onto an A-space Y, then Y has $\mathcal{P}$ if X has $\mathcal{P}$.

1.17. Theorems 1.12, 1.15 and 1.16, respectively, hold for properties mentioned in 1.4, 1.5 and 1.6. It may be noted that some properties are stronger that both isocompactness and ss-discreteness (for instance, metrizability, topological completeness, paracompactness etc.) and for those properties it will not be necessary to assume any condition on the domain space. For example, one can state the following result:

"Every closed continuous A-image of a metrizable space is metrizable".

1.18. Smith and Krajewski [16] talked of discrete HC- $\aleph_0$-expandable spaces (that is, every countable discrete family of closed sets is expandable to an H-closure-preserving collection of open sets) and proved that if $f:X \to Y$ is a closed continuous mapping from a discrete HC- $\aleph_0$-expandable space X onto a first - countable space Y, then Bd $f^{-1}(y)$ is countably compact for each $y \in Y$. Now every countably paracompact space is both ss-discrete and discrete HC- $\aleph_0$-expandable . However, we do not know the relationship between ss-discrete and discrete HC- $\aleph_0$-expandable. In any case, in view of the above result of Smith and Krajewski [16], Theorems 1.12 and 1.15 remain valid with "A-space" replaced by "first-countable" and "ss-discrete" replaced by "discrete HC- $\aleph_0$-expandable".

Let us now show that as applications of our theorems many known results are indeed improved.

1.19. Since every countably paracompact space is ss-discrete, we have the following result, in view of Theorem 1.12, which improves Theorem 3.3 of Zenor [19] which he proved with "first-countable" in place of "A-space" :

"If $f:X \to Y$ is a closed continuous mapping from a countably paracompact space $X$ onto an A-space $Y$, then $Y$ is countably paracompact".

1.20. Since every $\mathfrak{m}$-expandable or discretely expandable or $\mathfrak{m}$-paracompact space is ss-discrete, we have, in view of Theorem 1.12, the folowing result which improves Theorem 5.10 of Smith and Krajewski [16] which they proved for "first-countable spaces, in place of "A-spaces".

"If $f:X \to Y$ is a closed, continuous mapping from $X$ onto an A-space $Y$, then

(i) If $X$ is $\mathfrak{m}$-expandable, then $Y$ is $\mathfrak{m}$-expandable.
(ii) If $X$ is discretely expandable, then $Y$ is discretely expandable.
(iii) If $X$ is $\mathfrak{m}$-paracompact, then $Y$ is $\mathfrak{m}$-paracompact.

1.21. Since every M-space is ss-discrete and every normal quasi-perfect image of an M-space is an M-space, we have, in view of Theorems 1.10 and 1.12, the following result which improves a theorem of Ishii [6] :

"Let $X$ be an M-space. Then the following are equivalent for a closed continuous mapping from $X$ onto a normal space $Y$ :

(i) $Y$ is an M-space;
(ii) $Y$ is an A-space;
(iii) Bd $f^{-1}(y)$ is countably compact for each $y \in Y$."

The above result also improves Theorem 5.11 of Smith and Krajewski [16] .

1.22. Again, every wM-space is ss-discrete and hence, in view of Theorems 1.10 and 1.12, we have the following result which offers a significant improvement over Theorem 1.2 of Ishii [7] in as much as we do not even assume $X$ to be completely regular:

"Let $X$ be a wM-space. If $f:X \to Y$ is a closed continuous mapping from a wM-space $X$ onto a space $Y$, then the following are equivalent:

(i) $Y$ is a wM-space;
(ii) $Y$ is an A-space;
(iii) Bd $f^{-1}(y)$ is countably compact for each $y \in Y$.

1.23. We have already remarked that every topologically complete space is both isocompact and ss-discrete. Hence, in view of Theorem 1.14, we have the following result which improves Corollary 3.3 of Dykes [3] proved for "q-space" in place of "A-space":

"If $f:X \to Y$ is a closed continuous mapping of a topologically complete space X onto an A-space X, then Bd $f^{-1}(y)$ is compact for each $y \in Y$."

1.24. Since every real-compact space is ss-discrete and isocompact and since every weak cb , perfect image of a real-compact space is real-compact, therefore we have, in view of Theorem 2.15, the following result which improves Corollary 3.5 of Dykes proved for "a q-space" in place of "an A-space":

"If $f:X \to Y$ is a closed continuous mapping of a real-compact space X onto a weak cb, A-space Y, then Y is real-compact."

1.25. Since Čech-complete spaces are CH, preserved under perfect maps, ss-discrete and isocompact, we have the following result which improves Theorem 12 of Frolík [4] :

"If $f:X \to Y$ is a closed, continuous mapping from a metrizable, Čech-complete space X onto a space Y, then the following are equivalent:

(i) Y is metrizable;
(ii) Y is an A-space;
(iii) Bd $f^{-1}(y)$ is compact for each $y \in Y$.
(iv) Y is Čech-complete".

Let us close this section with one final remark.

1.26. The class of A-spaces is contained in the class of outer open A-spaces which we mentioned earlier in section 1. The concept of outer open A-spaces is the weakest modification of an A-space considered by Michael in [11] . It seems that amongst completely regular spaces and the spaces of non-measurable cardinality, the assumption of A-space in all our theorems can be further weakened to an outer open A-space,because the class of all A-spaces Y such that there is a closed continuous mapping from some space X onto Y with the property that Bd $f^{-1}(y)$ is countably compact for each $y \in Y$ is identical with the class of all outer open A-spaces with that property.This can be seen as follows: Every A-space is an outer open A-space. Now if $f:X \to Y$ is a closed continuous mapping from a space X onto an outer open A-space Y such that Bd $f^{-1}(y)$ is countably compact (and hence relatively pseudo-compact) for each $y \in Y$, then Y must be an A-space in view of a result of Michael [11].

## 2. The Closure-Preserving Sum Theorem

Let us denote the locally finite sum theorem by LFST and the closure-preserving sum theorem by CPST. We shall now prove a theorem which exhibits a nice connection of the CPST with the LFST in an A-space as we mentioned earlier.

Theorem 2.1. Let X be an A-space. Let $\mathcal{P}$ be a property such that (i) $\mathcal{P}$ is CH (ii) the LFST holds for $\mathcal{P}$. Then the CPST also holds for $\mathcal{P}$ in X.

Proof: Let $\{F_\alpha : \alpha \in \Lambda\}$ be an H-closure-preserving closed covering of X. Let $X^*$ be the disjoint topological sum of $F_\alpha$'s and suppose that $F^*_\alpha$'s are disjoint homeomorphic copies of $F_\alpha$'s. Let $P:X^* \to X$ be the natural projection map. Now, for each $y \in X$, Bd $P^{-1}(y)$ has at most one point in each $F^*_\alpha$. We shall prove that Bd $P^{-1}(y)$ must be finite for each $y \in X$. Suppose $\{x_n : n = 1,2,\dots\}$ is a countably infinite subset of Bd $P^{-1}(y)$. Let the $F^*_\alpha$ which contains $x_n$ be denoted by $U_n$. Now, if we repeat the proof of Theorem 1.10 with $x_n = y_n$, we shall arrive at a contradiction. Thus Bd $P^{-1}(y)$ must be finite for each $y \in X$. Now proceeding as in the proof of Theorem 1.1, we can have a closed subset Z of $X^*$ such that there is a finite-to-one closed continuous map f from Z onto $X^*$. Obviously, $\{Z \cap F^*_\alpha : \alpha \in \Lambda\}$ is a locally finite collection. Since finite-to-one closed continuous maps will take locally finite collections to locally finite collections, $\{f(Z \cap F^*_\alpha) : \alpha \in \Lambda\}$ is locally finite. Also, it is a closed refinement of $\{F_\alpha : \alpha \in \Lambda\}$. Since $\mathcal{P}$ is CH, $\{f(Z \cap F^*_\alpha) : \alpha \in \Lambda\}$ is a locally finite closed covering of X each member of which has $\mathcal{P}$. Then since the LFST holds for $\mathcal{P}$, X has $\mathcal{P}$. This completes the proof of the theorem.

In view of Theorem 4 of [1], we can now state the following:

Theorem 2.2. Let $\mathcal{P}$ be a property such that $\mathcal{P}$ is (i) CH (ii) preserved under disjoint sums (iii) preserved under finite-to-one closed continuous maps. If X is an A-space, then the CPST holds for

$\mathcal{P}$ in X.

2.3. There are several properties for which the CPST does not hold but for which the LFST holds. However, in view of Theorem 2.1 above, it follows that for all these properties the CPST will hold in an A-space. We list below some of these properties:

Regularity, metrizability, symmetrizability, the property of being an $\aleph$-space, the property of being a space of countable type, the property of being a $\Sigma$-space, the property of being a Čech-complete space, the property of being a normal M-space, the property of being a $\omega$M-space, the property of being an $M^*$-space, the property of being countably compact at infinity, the property of being paracompact of infinity, the property of being countably paracompact at infinity, the property of being normal at infinity, expandability, discrete-expandability, H.C. expandability, discrete H.C. expandability.

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