Toposym 4-B

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Products of $[a, b]$-chain compact spaces

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PRODUCTS OF [a,b] -CHAIN COMPACT SPACES

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We introduce here two notions of "chain compactness in an interval [a,b] of cardinal numbers," and state several results about products of such spaces. Our main result may be considered as a generalization to higher cardinals of the theorem of C.T. Scarborough and A.H. Stone [5, Theorem 5,5] which states that a product of no more than $\aleph_1$ sequentially compact spaces is countably compact. The complete proofs of these results* will appear elsewhere.

The concepts we introduce here are natural augmentations to the following two classical concepts of "compactness in an interval [a,b] of cardinal numbers."

Definition 1. (Alexandroff and Urysohn [1]). A space X is called [a,b] -compact in the sense of complete accumulation points (or $[a,b]^r$-compact) provided that if E is an infinite subset of X and if $|E|$ is a regular cardinal with $a \leq |E| \leq b$, then E has a complete accumulation point p in X (i.e., for every neighborhood $\mathcal{U}$ of p, we have $|\mathcal{U} \cap E| = |E|$).

Definition 2. (Yu. Smirnov [6]). A space X is called [a,b]-compact in the sense of open covers (or [a,b] -compact) provided that if $\mathcal{U}$ is an open cover of X with $a \leq |\mathcal{U}| \leq b$, then $\mathcal{U}$ has a subcover $\mathcal{U}'$ with $|\mathcal{U}'| \leq a$.

For a discussion of these concepts, we refer the reader to [7] and [8].

Definition 3. A net [3, Chapter 2] $f: W \rightarrow X$ with a well-ordered domain is called a transfinite sequence, and is said to have a convergent subsequence if there exists a cofinal subset $A \subset W$ such that $f|_A : A \rightarrow X$ converges to a point in X.

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<u>Definition 4.</u> A space is called $[a,b]$ -<u>chain compact</u> (resp. $[a,b]^r$ -chain compact) if for every cardinal m in $[a,b]$ (resp. for every regular cardinal m in $[a,b]$) every transfinite sequence $f:m \to X$ has a convergent subsequence. A space in which every transfinite sequence has a convergent subsequence is called <u>chain compact.</u>

In this terminology, a space is sequentially compact if and only if it is $[\omega, \omega]$ -chain compact. It is also known [4, Theorem 4] that a space is chain compact if and only if it is compact and scattered.

It is easy to see that a finite product of $[a,b]$ -chain compact (resp. $[a,b]^r$-chain compact) spaces is $[a,b]$ -chain (resp. $[a,b]^r$-chain compact). Concerning infinite products we have the following two results:

<u>Theorem 1.</u> A countable product of $[a,b]$ -chain compact (resp. $[a,b]^r$-chain compact) spaces is $[a,b]$ -compact (resp. $[a,b]^r$-compact).

<u>Theorem 2.</u> A product of no more than $\aleph_1$ $[\omega, b]$ -chain compact spaces is $[\omega, b]$ -complet.

<u>Corollary</u> (Scarborough - Stone). A product of no more that $\aleph_1$ sequentially compact spaces is countably compact.

We now outline how Theorems 1 and 2 may be proved as easy corollaries of a general product theorem (Lemma 3 below).

Let Φ be a class of filter bases. A filter base $\mathcal{F}$ on a set X is called a Φ-<u>filter base</u> if $\mathcal{F} \in \Phi$. A space X is called Φ-<u>compact</u> if every Φ-filter base $\mathcal{F}$ on X has an adherent point (i.e., $\cap \{\bar{F}:F \in \mathcal{F}\} \neq \phi$). A filter base is called <u>total</u> if each finer filter base has an adherent point, and a space X is called <u>totally</u> Φ-<u>compact</u> if every Φ-filter base on X has a finer, total, Φ-filter base. These definitions are discussed more fully in [9] and [10] . Here are some examples of Φ-compactness used in this paper.

1. Let Φ_m denote the class of all filter bases G which have a base $\mathcal{F} = \{F_\alpha : \alpha < m\}$ such that if $\alpha < \beta < m$, then $F_\alpha \supset F_\beta$ .Clearly, Φ_ω-compactness is equivalent to countable compactness.

2. Let $\Phi_{m \times \omega}$ denote the class of all filter bases G which have a base $\mathcal{F} = \{F(\alpha, n): \alpha < m$ and $n < \omega\}$ such that if $\alpha \leq \alpha'$ and $n \leq n'$ then $F(\alpha, n) \supset F(\alpha', n')$.

Total Φ_ω-compactness is called <u>total countable compactness.</u> For $T_{3\frac{1}{2}}$ -spaces, a space is totaly countably compact if and only if it is a member of Z. Frolík's class P_F [2].

<u>Lemma 1.</u> (a). If X is $[m,m]$-chain compact, then X is totally Φ_m-compact. In particular, every sequentially compact space is totally countably compact.

(b). If X is sequentially compact and $[m,m]$-chain compact, then X is totally $\Phi_{m\times\omega}$-compact.

A class Φ of filter bases is said to be $<m$-additive provided that if $\{\mathcal{F}_\alpha : \alpha \in A\}$ is a family of Φ-filter bases on a set X, and $|A| < m$, then $\sup\{\mathcal{F}_\alpha : \alpha \in A\} \in \Phi$ if it exists, where $\sup\{\mathcal{F}_\alpha : \alpha \in A\}$ is the set of all finite intersections from $\bigcup\{\mathcal{F}_\alpha : \alpha \in A\}$ provided all such intersections are non-empty. A class Φ is said to be <u>stable under functions</u> (resp. <u>inverse functions</u>) provided that for every function $f: X \to Y$, if $\mathcal{F}$ is a Φ-filter base on X, then $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\} \in \Phi$ (resp. if $\mathcal{F}$ is a Φ-filter base on $f(X) \subset Y$, then $f^{-1}(\mathcal{F}) = \{f^{-1}(F) : F \in \mathcal{F}\} \in \Phi$).

<u>Lemma 2.</u> (a). For all m, Φ_m is finitely additive (i.e., $<\omega$-additive) and Φ_m is countably additive (i.e., $<\omega_1$-additive) if and only if $cf(m) = \omega$.

(b). For all m, $\Phi_{m\times\omega}$ is countably additive, but not $<\omega_2$-additive.

(c). Both classes Φ_m and $\Phi_{m\times\omega}$ are stable under functions and inverse functions.

The remaining result which we need is a corollary to Theorem 1 of [10].

<u>Lemma 3.</u> Let Φ be a class of filter bases which is stable under functions and inverse functions. Assume that Φ is $<k$-additive, where k is an infinite cardinal number. If $\{X_\alpha : \alpha < k\}$ is a family of totally Φ-compact spaces, then $\Pi\{X_\alpha : \alpha < k\}$ is Φ-compact.

To prove Theorem 2, let $\{X_\alpha : \alpha < \omega_1\}$ be a family of $[\omega,m]$-chain compact spaces. For each infinite cardinal number $n \leq m$, the spaces X_α are totally $\Phi_{n\times\omega}$-compact. Thus $X = \Pi\{X_\alpha : \alpha < \omega\}$ is $\Phi_{n\times\omega}$-compact, hence $[n,n]$-compact for all $n \leq m$, thus X is $[\omega,m]$-compact.

Theorem 1 is proved in a similar manner. Other applications of Lemma 3 are given in [10].

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