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ON PRESERVING THE FIXED POINT PROPERTY BY MAPPINGS

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Let a metric compactum Y be a continuous image of a compactum X under a mapping $p:X \to Y$. The question arises when the fixed point property of the space Y can be infered from the fixed point property of the space X? It is clear that each mapping $f:Y \to Y$ (multi-valued mapping $F:Y \to Y$) of the space Y into itself generates a multi-valued mapping $G = p^{-1} \circ f \circ p$ $(G = p^{-1} \circ F \circ p)$ of the space X into itself. Therefore, if the space X has the fixed point property with respect to a class of mappings containing the mapping G, then the mapping f (F) has a fixed point.

1. Let $F:Z \to T$ be a multi-valued mapping. We say that the mapping F satisfies the Vietoris condition (V), if for each point $z \in Z$ the set $F(z)$ is acyclic (the reduced Vietoris homology groups with rationals Q as coefficients are regarded). Further all multi-valued mappings are assumed to be upper semi-continuous and all single-valued mappings are assumed to be continuous.

Proposition 1. If the compactum X has the fixed point property with respect to finite compositions of multi-valued mappings satisfying the Vietoris condition (V), then each acyclic continuous image Y of the compactum X has this property.

Really, if $F = F_n \circ \ldots \circ F_1$ where F_i, $i = 1, \ldots, n$, satisfies the condition (V), then $G = p^{-1} \circ F \circ p = p^{-1} \circ F_n \circ \ldots \circ F_1 \circ p$ is a finite composition of mappings satisfying the condition (V). Hence there exists a point $x \in X$ such that $G(x) = (p^{-1} \circ F \circ p)(x) \ni x$. Then for the point $y = p(x)$ we have $(p^{-1} \circ F)(y) \ni x$ or $p((p^{-1} \circ F)(y)) = F(y) \ni p(x) = y$. Consequently y is a fixed point of the mapping F.

If the Lefschetz number $\Lambda(F)$ of the multi-valued mapping F is defined (for example, if the mapping F is a finite superposition of mappings satisfying the condition (V) and the space Y has a finitely generated homology, that is, all homology groups $H_n(Y;Q)$ are finitely generated and for all sufficiently large n the homology groups $H_n(Y;Q)$ are vanishing), then the Lefschetz number is defined for the mapping G as well, the equality $\Lambda(G) = \Lambda(F)$ being true. Therefore, everything said above about the "multi-valued fixed point property" is justified for the "property of satisfying the multi-valued Lefschetz theorem" as well. As p is a single-valued mapping, both $p^{-1}\circ F_n$ and $F_1 \circ p$ satisfy the condition (V) and hence G is a superposition of n mappings satisfying the condition (V) as well. In particular, we can state

Proposition 2. Let the inverse images of the points $y \in Y$ under the mapping $p: X \to Y$ be acyclic and let X be an $M\Lambda$-space in the sense of Powers [13]. Then Y is an $M\Lambda$-space as well.

Corollary 1. Let the inverse images of the points $y \in Y$ under the mapping $p: X \to Y$ be acyclic. Let X be an approximative absolute neighbourhood retract in the sense of Noguchi [12]. Then Y is an $M\Lambda$-space.

In 1946, Eilenberg and Montgomery [7] proved the following coincidence theorem. Let M be an absolute neighbourhood retract, N a compact metric space and let $r: N \to M$, $t: N \to M$ be continuous mappings such that t^{-1} satisfies the condition (V). Consider the Lefschetz number $\Lambda(r,t) = \Sigma(-1)^i$ trace $(r_i^* t_i^{-1*})$. If $\Lambda(r,t) \neq 0$ then r and t have a coincidence.

Proposition 3. For a compactum M with a finitely generated homology, the following conditions are equivalent:

1) for the space M the Eilenberg-Montgomery theorem is true;
2) for the space M the Lefschetz fixed point theorem is true for multi-valued mappings F possessing the following representation: $F = r \circ t^{-1}$, where the mapping t^{-1} satisfies the condition (V) and is inverse to the single-valued continuous mapping while r is a single-valued continuous mapping;
3) for the space M the Lefschetz fixed point theorem is true for compositions of multi-valued mappings, satisfying the condition (V) (that is, for the space M the Lefschetz fixed point theorem is true for mappings admissible in the sense of Powers [14]);
4) for the space M the Eilenberg-Montgomery coincidence theorem is true for multi-valued mappings $t: N \to M$ and $r: N \to M$ such that both t^{-1} and r satisfy the condition (V) (here

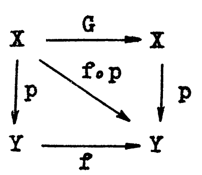
a coincidence point means a point $x \in N$ such that $r(x) \cap t(x) \neq \emptyset$);

5) for every collection of single-valued mappings $f_1 : Y_1 \to M$, $g_n : Y_n \to M$, $f_{i+1} : Y_{i+1} \to X_i$ and $g_i : Y_i \to X_i$, where X_i and Y_i, $i=1,\dots,n-1$, Y_n are arbitrary compacta and the mappings f_i^{-1}, $i=1,\dots,n$, satisfy the condition (V), the Lefschetz number
$$\Lambda(g_n, f_n, \dots, g_1, f_1) = \Lambda(g_n f_n^{-1} \dots g_1 f_1^{-1}) =$$
$$= \sum (-1)^i \operatorname{trace} (g_n^* \circ f_n^{-1*} \circ \dots \circ g_1^* \circ f_1^{-1*})$$
is determined and, provided $\Lambda(g_n, f_n, \dots, g_1, f_1) \neq 0$, there exist points $y_1, \dots, y_n$, $y_i \in Y_i$, such that $f_1(y_1) = g_n(y_n)$ and $f_{i+1}(y_{i+1}) = g_i(y_i)$ for $i=1,\dots,n-1$;

6) for every collection of multi-valued mappings $f_1 : Y_1 \to M$, $g_n : Y_n \to M$, $f_{i+1} : Y_{i+1} \to X_i$ and $g_i : Y_i \to X_i$, where X_i and Y_i, $i=1,\dots,n-1$, Y_n are arbitrary compacta and the mappings f_i^{-1}, g_i, $i=1,\dots,n$ satisfy the condition (V), the Lefschetz number
$$\Lambda(g_n, f_n, \dots, g_1, f_1) = \Lambda(g_n f_n^{-1} \dots g_1 f_1^{-1}) =$$
$$= \sum (-1)^i \operatorname{trace} (g_n^* \circ f_n^{-1*} \dots \circ g_1^* \circ f_1^{-1*})$$
is determined and, provided $\Lambda(g_n, f_n, \dots, g_1, f_1) \neq 0$, there exist points $y_1, \dots, y_n$, $y_i \in Y_i$, such that $f_1(y_1) \cap g_n(y_n) \neq \emptyset$ and $f_{i+1}(y_{n+1}) \cap g_i(y_i) \neq \emptyset$ for $i = 1,\dots,n-1$.

As the author has learnt at the Symposium, analogous results were also obtained by M. van de Vel.

2. Let us consider a mapping $G = p^{-1} \circ f \circ p$, where $f : Y \to Y$ is a single-valued mapping. The mapping G is not supposed to have an exact selection or equivalently, there are examples where the mapping $f \circ p$ cannot be lifted with respect to the projection p. Nonetheless, in some cases the mapping G possesses an approximation g such that the existence of a fixed point for the mapping g implies the existence of a fixed point for the mapping f. However, for this we are forced to require from the mapping p some stronger conditions of acyclicity.



Definition [3]. A closed subset A of a space X is said to be approximatively connected in X in all dimensions not exceeding n ($A \in AC_X^n$) if for each neighbourhood U of the set A in X there exists a neighbourhood V of A in X such that every mapping $h:S^k \to V$ of a k-dimensional space, where $k \le n$, is homotopic to 0 in U.

The above mentioned almost selection of the mapping G (almost lifting of the mapping $f \circ p$) is obtained by the following theorem, proved in slightly different forms by many authors [1,2,6,9,10,11].

Theorem. Let $p:X \to Y$ be a mapping of a compactum X onto a compactum Y such that $p^{-1}(y) \in AC_X^n$ for all points $y \in Y$. Then $Y \in LC^n$ and for each mapping $h:Z \to Y$, where Z is a compactum of a dimension not greater than $n+1$, and every $\varepsilon > 0$ there exists a mapping $g:Z \to X$ such that $\rho(pg,h) < \varepsilon$.

This theorem implies easily enough the following fixed point theorems.

We shall write $X \in fpp$, if each mapping $g:X \to X$ of the space X into itself possesses a fixed point.

Theorem 1. Let $p:X \to Y$ be a mapping of a compactum $X \in fpp$ onto a compactum Y such that $p^{-1}(y) \in AC_X^n$ for all points $y \in Y$. Then $Y \in fpp$ provided $\dim X \le n+1$ or $\dim Y \le n+1$. Moreover, in the general case the space Y possesses the following property: every mapping $f:Y \to Y$ such that $\dim f(Y) \le n$ has a fixed point.

This theorem was proved by J. Cobb and W. Voxman [5] in the following cases: 1) X is an n+1-dimensional polyhedron, 2) Y is embeddable into R^{n+1}.

A closed subset A of an ANR-compactum X is an AC_X^0-subset iff A is connected [3].

Consequently, from Theorem 1 we obtain the following

Corollary 2. If the space Y is a monotonous image of an AR-compactum X, then for every mapping $f:Y \to Y$ such that $\dim f(Y) = 1$ there exists a fixed point.

An analogous theorem in the infinite-dimensonal case is also true.

Theorem 2. Let $p:X \to Y$ be a mapping of a compactum $X \in fpp$ onto a compactum Y such that $p^{-1}(y) \in AC^n_X$ for all integers n and all points $y \in Y$. Then $Y \in fpp$ in each of the following cases: 1) $\dim X$ or $\dim Y$ is finite; 2) X or Y is a product of finite dimensional compacta; 3) X or Y is an approximative absolute neighbourhood retract in the sense of Clapp ($AANR_c$) [4] .

In the above theorems we have conditions on the embedding of the inverse images $p^{-1}(y)$ of the points $y \in Y$. It is clear that these conditions can be avoided if any kind of simplicity of the local structure of the space X is supposed, for example if $X \in LC^n$, $X \in LC^\infty$ or $X \in ANR$. But it turns out that the absolute properties upon the inverse images of points can be required also in some other cases.

Definition. We shall write $A \in AC^n$ if there is an embedding of A into an ANR-compactum X such that $A \in AC^n_X$.

Theorem 3. Let $p:X \to Y$ be a mapping of an $AANR_c$-compactum $X \in fpp$ onto a compactum Y such that $p^{-1}(y) \in AC^n$ for all points $y \in Y$. Then Y has the fixed point property if $\dim X \leq n+1$ or $\dim Y \leq n+1$.

Theorem 4. Let $p:X \to Y$ be a mapping of an $AANR_c$-compactum X onto a compactum Y such that $p^{-1}(y) \in AC^\infty$ for all $y \in Y$. Then if X has the fixed point property, so does Y.

3. The basic method for proving Theorems 1-4 consists in finding a mapping $g:X \to X$ such that the mappings $f \circ p$ and $p \circ g$ are near to each other. This easily implies that for each N and each $\varepsilon > 0$ there exists a mapping $g:X \to X$ such that $\rho(f^k p, pg^k) < \varepsilon$ for all $k \leq N$. It is evident from this that Theorems 1-4 of Section 2 remain true if we replace the fixed point property by the following property: "there exists an N such that for every mapping $g:X \to X$ the mapping $g^N:X \to X$ has a fixed point".

Let us remark also that for the space X the following conditions are equivalent: 1) there exists N such that for every mapping $g:X \to X$ the mapping $g^N:X \to X$ has a fixed point; 2) there exists N^* such that for each mapping $g:X \to X$ the mapping $g^k:X \to X$ for some $k \leq N^*$ dependent on g has a fixed point. It is easy to verify that if there exists such N then we must put $N^* = N$ and if there exists such N^* then we must put $N = N^*!$

4. In [15] Sieklucki introduced the notion of the quasi-deformation retract and proved that a quasi-deformation retract of an AR-compactum has the fixed proint property. Let us formulate one theorem about the behaviour of quasi-deformation retracts under the cell-like mappings.

Theorem 5. Let $p:X \to Y$ be a mapping of a compactum X onto a finite-dimensional compactum Y such that $p^{-1}(y) \in AC^{\infty}$ for every point $y \in Y$. Then if X is a quasi-deformation retract of a finite-dimensional AR-compactum, so is Y.

Let $p:X \to Y$ and $q:T \to Z$ be mappings such that $p^{-1}(y) \in AC^n_X$ and $q^{-1}(z) \in AC^n_T$ for all points $y \in Y$ and $z \in Z$, respectively. Then for the mapping $p \times g:X \times T \to Y \times Z$ we have $(p \times q)^{-1}(y,z) = (p^{-1})(y) \times (q^{-1})(z) \in AC^n_{X \times T}$ for $(y,z) \in Y \times Z$ and consequently the following is true : if $p:X \to Y$ is a cell-like mapping of an $AANR_c$-compactum X onto Y and $X \times I$ has the fixed point property, then the space $Y \times I$ also has the fixed point property.

This result is of interest for there are contractible continua X with the fixed point property such that $X \times I$ has not the fixed point property (such continuum X cannot be an $AANR_c$-compactum) and there are simply connected polyhedra X with the fixed point property such that $X \times I$ has not the fixed point property (such an $AANR_c$-compactum cannot be contractible) [8] . Let us note that if X is a quasi-deformation retract of an AR-compactum, then the space $X \times I$ is also a quasi-deformation retract of an AR-compactum and hence it has the fixed point property [15] .

5. If the compactum Y has a finitely generated homology,then there exists $\varepsilon > 0$ such that two ε -near mappings f and $g:X \to Y$ generate the same homomorphism of the homology. Hence $(fp)^* = f^* \circ p^* = (pg)^* = p^* \circ g^*$. Since p^* is an isomorphism of the homology groups, the Lefschetz numbers $\Lambda(f)$ and $\Lambda(g)$ of the mappings f and g are equal. From this, it is evident that everything said in Sections 2 and 3 concerning the "absolute" fixed point property remains true with respect to the property of satisfying the Lefschetz fixed point theorem. In particular, the multi-valued mapping G has an approximation $g:X \to X$ such that the Fuller indices $\Phi(g)$ and $\Phi(f)$ of the mappings g and f respectively are equal. Consequently, the validity of the Fuller theorem [8] in the space X implies the validity of the Fuller theorem in the space Y .

The author has learnt at the Symposium that P.Minc proved the Lefschetz fixed point theorem for quasi-deformation retracts of ANR-compacta. Hence the following theorem is a theorem about the behaviour of the fixed point property.

Theorem 6. Let $p:X \to Y$ be a mapping of a compactum X onto a finite-dimensional compactum Y such that $p^{-1}(y) \in AC^{\infty}$ for every point $y \in Y$. Then if X is a quasi-deformation retract of a finite-dimensional ANR-compactum, so is Y.

6. We have considered only the metric compacta, but some of the results formulated here are true under more general assumptions. Let us close with a result concerning the fixed point property for an important class of spaces including both compact and metric spaces. This is the class of p-paracompacta in the sense of Archangelskij, which consists of spaces admitting a perfect mapping onto a metric space.

Theorem 7. Let a p-paracompactum X be an absolute neighbourhood retract in the class of p-paracompacta. Then for a compact mapping $g:X \to X$ of the space X into itself, one can define the Lefschetz number $\Lambda(g)$ and the mapping g has a fixed point if this Lefschetz number is different from 0.

From this theorem one can derive generalizations of some above formulated theorems.

REBERENCES

[1] S.Armentrout, T.Price: Decompositions into compact sets with UV properties. Trans. Amer. Math. Soc. 141 (1969), 433-442.

[2] S.Bogatyi: O teoreme Vietorisa v kategorii sheypov i ob odnoy probleme Borsuka. Fund. Math. 84 (1974), 209-228.

[3] K.Borsuk: A note on the theory of shape of compacta. Fund. Math. 67 (1970), 265-278.

[4] M.H.Clapp: On a generalization of absolute neighbourhood retracts. Fund. Math. 70 (1971), 117-130.

[5] J.Dugundji: Modified Vietoris theorems for homotopy. Fund. Math. 66 (1970), 223-235.

[6] J.Cobb, W.Voxman: Some fixed point results for UV decompositions of compact metric spaces. Proc.Amer.Math.Soc. 33 (1972),156-160.

[7] S.Eilenberg, D.Montgomery: Fixed point theorems for multivalued transformations. Amer. J. Math. 68 (1946), 214-222.

[8] E.Fadell: Recent results in the fixed point theory of continuous maps. Bull. Amer. Math. Soc. 76 (1970), 10-29.

[9] V.P.Kompanietz: Gomotopicheskij kriterij tochechnogo otobragenija. Ukrain. Mat. Ž. 18 (1966), 3-10.

[10] G.Kozlowski: Factorization of certain maps up to homotopy. Proc. Amer. Math. Soc. 21 (1969), 88-92.

[11] R.C.Lachner: Cell-like mappings, I. Pacific J. Math. 30 (1969), 717-731.

[12] H.Noguchi: A generalization of absolute neighborhood retracts. Kōdai Math. Sem. Rep. No 1 (1953), 20-22.

[13] M.Powers: Multi-valued mappings and Lefschetz fixed point theorems. Proc. Camb. Phil. Soc. 68 (1970), 619-630.

[14] M.Powers: Lefschetz fixed point theorems for a new class of multi-valued maps. Pacific J. Math. 42 (1973), 211-220.

[15] K.Sieklucki: Generalized retractions and fixed points. Bull.Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), 945-949.

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