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Liftings of functors in topological situations

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LIFTINGS OF FUNCTORS IN TOPOLOGICAL SITUATIONS

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We are going to deal with liftings of a functor $P : X \to X$ along a functor $V : B \to X$, i.e. with functors $S : B \to B$ such that

(1) $$VS = PV.$$

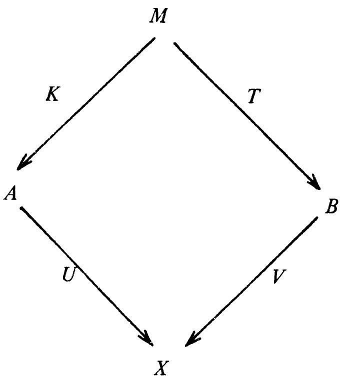
This problem is a special case of a more general problem of the search of functors $S : A \to B$ such that

(2) $$VS = U$$

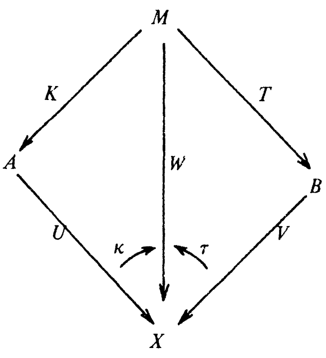
where $U : A \to X$ is a given functor. Further, we will consider a monad (P,η,μ) in X and we will be interested in liftings of a monad (P,η,μ) along V, i.e. in monads $(S,\bar{\eta},\bar{\mu})$ in B such that $VS = PV$, $V\bar{\eta} = \eta V$ and $V\bar{\mu} = \mu V$.

We will try to carry over the investigation of functors S to the study of their restrictions T on a suitable full subcategory M of B. So we are going to consider the following diagram

(3)



and we will search functors $S : A \to B$ making the both triangles commute. We will emphasize the case when V is an $(\mathscr{E}, \mathscr{M})$-topological functor in the sense of H.Herrlich [2]. Then the corresponding problem (3) is solved by the construction of M.Hušek (see [3]). But for the study of liftings of monads or for liftings of functors along a non-topological V it is necessary to work in a more general situation than (3) and namely in

(4) 

Here, $\kappa : UK \to W$ and $\tau : VT \to W$ are natural transformations. This situation is introduced and dealt with by R.Guitart in [1] . If $UK = W = VT$ and $\kappa = 1_W = \tau$, we get (3).

Our results give a general point of view to the investigation of M.Sekanina (see this Proceeding). I would like to express my gratitude to him for turning my interest to these questions and for many valuable discussions. Details and relevant considerations can be found in the author's paper "Extensions of functors and their applications" which will appear elsewhere. Concerning concepts of the category theory see [4] .

1. Extensions of functors

1.1. Definition. Consider (3) . A couple $(R,\overline{\rho})$ consisting of a functor $R : A \to B$ and a natural transformation $\overline{\rho} : RK \to T$ is called a *right extension* of T along K if $VR = U$, $V\overline{\rho} = 1_{VT}$ and for any $S : A \to B$ and $\overline{\sigma} : SK \to T$ such that $VS = U$ and $V\overline{\sigma} = 1_{VT}$ there is a unique natural transformation $\alpha : S \to R$ such that $V\alpha = 1_U$ and $\overline{\sigma} = \overline{\rho} \cdot \alpha K$.

1.2. Definition. Consider (4) . A triple $(R,\rho,\overline{\rho})$ consisting of a functor $R : A \to B$ and natural transformations $\rho : VR \to U$, $\overline{\rho} : RK \to T$ is called a *lax right extension* of (T,τ) along (K,κ) if $\kappa \cdot \rho K = \tau \cdot V\overline{\rho}$ and for any $S : A \to B$, $\sigma : VS \to U$ and $\overline{\sigma}$: : $SK \to T$ such that $\kappa \cdot \sigma K = \tau \cdot V\overline{\sigma}$ there is a unique natural transformation $\alpha : S \to R$ such that $\sigma = \rho \cdot V\alpha$ and $\overline{\sigma} = \overline{\rho} \cdot \alpha K$.

Clearly if $(R, 1_U, \overline{\rho})$ is a lax right extension of $(T, 1_W)$ along $(K, 1_W)$, then $(R,\overline{\rho})$ is a right extension of T along K .

To avoid the complications with natural isomorphisms we will suppose that V is amnestic (i.e. any isomorphism f of B such that $Vf = 1$ is the identity) and it has the property of transfer (i.e. for any isomorphism $g : Vb \to x$ in X there is an isomorphism f of B such that $Vf = g$) . In what follows, to simplify (4) we will assume that V is faithful, $VT = W$ and $\tau = 1_W$. Remark however that the following generalization of the Hušek's construction can be carried out in a general case, too.

1.3. Construction. Suppose that for every $a \in A$ there exist $Ra \in B$ and $\rho_a : VRa \to Ua$ such that for any $f : a \to Km$, where $m \in M$, there is $\widetilde{f} : Ra \to Tm$ such that $\kappa_m \cdot Uf \cdot \rho_a = V\widetilde{f}$ and moreover Ra, ρ_a have the following universal property : For any $b \in B$ and $u : Vb \to Ua$ such that for any $f : a \to Km$ there is $f' : b \to Tm$ such that $\kappa_m \cdot Uf \cdot u = Vf'$ there exists a unique morphism $t : b \to Ra$ such that $\rho_a \cdot Vt = u$.

Then $R : A \to B$ is a functor and $\rho : VR \to U$ a natural transformation. Further, the equality $\rho_m = \widetilde{1}_{Km}$ defines a natural transformation $\overline{\rho} : RK \to T$.

1.4. Theorem. $(R,\rho,\overline{\rho})$ from 1.3. is a lax right extension of (T,τ) along (K,κ) . If K is full and κ an isomorphism, then $\overline{\rho} = 1_T$.

1.5. Theorem. Consider (3) . Let V be $(\mathscr{E},\mathscr{M})$-topological and $\langle Uf \backslash f : a \to Km, m \in M \rbrace \in \mathscr{M}$ for any $a \in A$. Then 1.3. provides a right extension $(R,\overline{\rho})$ of T along K.

Let $E(U,V)$ be the class of all functors $S : A \to B$ such that $VS = U$ ordered by means of : $S \leqslant S'$ iff there is a natural transformation $\alpha : S \to S'$ such that $V\alpha = 1_U$. Then the composition with K gives an isotone map $E(K,V) : E(U,V) \to E(UK,V)$.

1.6. Corollary. Let V be $(\mathscr{E},\mathscr{M})$-topological, $\langle Uf \backslash f : a \to Km , m \in M \rbrace \in \mathscr{M}$ for every $a \in A$ and K be full.

Then $E(K,V)$ is onto and it induces a bijection between maximal elements of $E(U,V)$ and $E(UK,V)$.

This result shows that the insight into functors $T : M \to B$ such that $VT = UK$ enables us to find maximal elements of $E(U,V)$.

1.7. Example. Let Ord be the category of all ordered sets and $V : \mathrm{Ord} \to \mathrm{Set}$ be the forgetful functor. Let $P^+ : \mathrm{Set} \to \mathrm{Set}$ be the covariant power set functor. Let M be a full subcategory of Ord having the three-element chain 3 as the only object and $K : M \to \mathrm{Ord}$ be the inclusion. The functor V is (extremally epi-monosource)-topological and it is easy to show that $\langle P^+ Vf \backslash f : b \to 3 \}$ is a monosource for any ordered set b. By 1.6. $E(K,V)$ induces a bijection between maximal elements of $E(U,V)$ and $E(UK,V)$. Hence, there are only finitely many maximal elements of $E(U,V)$. These maximal elements are dealt with in [6].

In a general case construction 1.3. does not yield a right extension. But it can be given a construction of a right extension from a lax one which was introduced in a special case in [5].

2. Liftings of monads

In what follows, let (P,η,μ) be a monad in X.

2.1. Definition. If $(S,\overline{\eta},\overline{\mu})$ is a monad in B and $\sigma : VS \to PV$ a natural transformation such that $\sigma \cdot V\overline{\eta} = \eta V$ and $\sigma \cdot V\overline{\mu} = \mu V \circ P\sigma \cdot \sigma S$, then we say that (S,σ) (more precisely $((S,\overline{\eta},\overline{\mu}),\sigma)$) is a *lax lifting* of a monad (P,η,μ) along V.

Morphisms of liftings $\alpha : (S,\sigma) \to (S',\sigma')$ are taken as morphisms of monads $\alpha : S \to S'$ such that $\sigma' \cdot V\alpha = \sigma$. In this way we get a category $Z(P,V)$ of lax liftings of a monad P along V.

Liftings of (P,η,μ) along V are lax liftings $(S,1_{PV})$. Denote by $Z_E(P,V)$ the full subcategory of $Z(P,V)$ constisting of all liftings. $Z_E(P,V)$ is a full subcategory of $E(PV,V)$.

Consider (3) for $U = PV$ and construct the following special case of (4)[(1)]. Here $M + M$ is the sum of two copies of a category M and $[K,T]$, $\cdots$, $[1_{VT},1_{VT}]$ are induced functors and natural transformations.

Denote by $Z(P,V,T)$ a subcategory of $Z(P,V)$ consisting of all lax liftings (S,σ) such that $SK = T$ and $\sigma K = 1_{VT}$ and of morphisms of liftings α such that $\alpha K = 1_T$. Further, $Z_E(P,V,T)$ will be a full subcategory of $Z(P,V,T)$ consisting of all liftings S such that $SK = T$.

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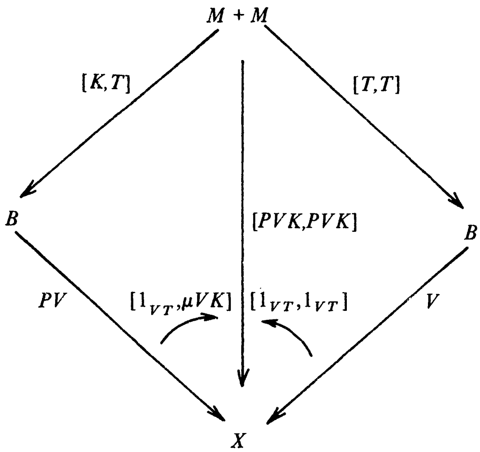
(1) Next page.

(5)



2.2. Theorem. Let K be full and $\nu : K \to T$ a natural transformation such that $\eta VK = V\nu$. Let $(R,\rho,[\rho_0,\rho_1])$ be a lax right extension of $([T,T]\ ,\ [1_{VT},1_{VT}])$ along $([K,T]\ ,\ [1_{VT},\mu_{VK}])$ and let it will be provided by 1.3.

Then there exist natural transformations $\tilde{\eta},\tilde{\mu}$ such that $(R,\tilde{\eta},\tilde{\mu})$ is a monad , (R,ρ) a lax lifting of P along V and the following conditions are equivalent:

(i) $Z(P,V,T) \neq \phi$

(ii) $(R,\rho) \in Z(P,V,T)$

(iii) For any $m,n \in M$ and any $g : Kn \to Tm$ there exists $\overline{g} : Tn \to Tm$ such that $V\overline{g} = \mu_{VKm} \cdot PVg$.

If these equivalent conditions are satisfied, then (R,ρ) is a terminal object of $Z(P,V,T)$.

2.3. Corollary. Let K be full, V $(\mathcal{E},\mathcal{M})$-topological and $\{ PVf\backslash f : a \to Km,\ m \in M\} \in \mathcal{M}$ for any $a \in A$.

Then $Z_E(P,V,T) \neq \phi$ iff there is a natural transformation $\nu : K \to T$ such that $V\nu = \eta VK$ and it holds the condition (iii) from 2.2. In this case $(R,\tilde{\eta},\tilde{\mu})$ from 2.2. is the greatest element of $Z_E(P,V,T)$.

This result characterizes restrictions T of liftings of P and shows that R's constructed for maximal-restrictions T's are maximal elements in $Z_E(P,V)$. However (5) is not functorial in T, so 2.3. admits the case that a maximal R induces a T which is not maximal.

Liftings of a monad P^+ on Ord was dealt with by M.Sekanina. 2.3. implies that there are only finitely many such maximal liftings.

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