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EXTENSION OF MEASURES AND INTEGRALS BY THE HELP OF A PSEUDOMETRIC

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0. Introduction. There are two main concepts in the measure theory. The measure can be regarded as a set function defined on a set of subsets of a given set. On the other hand measure can be regarded as a functional defined on a set of real-valued functions. In both concepts an extension process is necessary.

In this communication we present a common generalization of both concepts. We study a real-valued function J_0 defined on a sublattice A of a given lattice H with some properties. If we define a suitable pseudometric, then J_0 becomes a uniformly continuous function, it can be extended to the closure A^- of A and this is the requested extension.

If H is a suitable lattice of sets, then the measure extension theorem is obtained. If H is a suitable lattice of real-valued functions, then the extension theorem for Daniell integrals (or Radon measures) is obtained.

Our extension process consists of the following three steps.

1. To a given sublattice A of H and a mapping $J_0 : A \to R$ we construct a mapping $J : H \longrightarrow R$ extending J_0.

In this step H is assumed to satisfy the following conditions: H is boundedly σ-complete, σ-continuous lattice and to every $x \in H$ there are $a_n \in A$ such that $x \leqq \bigvee a_n$. The initial mapping J_0 is increasing, J_0 is a valuation (i.e. $J_0(a) + J_0(b) = J_0(a \vee b) + J_0(a \wedge b)$) and J_0 is upper continuous (i.e. $x_n \in A$, $x \in A$, $x_n \nearrow x \Rightarrow J_0(x_n) \to J_0(x)$).

Put $A^+ = \{ b \in H ; \exists\, a_n \in A , a_n \nearrow b \}$, $J^+ : A^+ \to \bar{R}$, $J^+(b) = \lim J_0(a_n)$. (Under previous assumptions this limit does not depend on the choice of a_n.) Finally $J^*(x) = \inf \{ J^+(b) ; b \geqq x , b \in A^+ \}$.

J^* has also some nice properties, e.g. J^* is upper continuous on H.

2. In the second step we assume that there are given three binary operations $\triangle$, $+$, $\setminus$: $H \times H \to H$ satisfying some conditions. In the set lattice case, $A \triangle B$ is the symmetric difference, $A \setminus B$ is the difference and $A + B$ is the union of the sets A, B. In the function lattice case, $f \triangle g(x) = |f(x) - g(x)|$, $f \setminus g(x) = f(x) - \min(f(x), g(x))$, $f + g(x) = f(x) + g(x)$.

We use the following properties of the algebraic structure: H has the least element 0 contained in A, A is closed under $\triangle$, $\setminus$, $+$; $a \triangle a = 0$, $a \triangle 0 = a$, $a \triangle b = b \triangle a$, $a + b = b + a$, $a \triangle b \leqq$

$\leqq (a \triangle c) + (b \triangle c)$, $(a \vee b) \triangle (c \vee d) \leqq (a \triangle c) + (b \triangle d)$, $(a \wedge b) \triangle (c \wedge d) \leqq (a \triangle c) + (b \triangle d)$, $(a + b) \triangle (c + d) \leqq (a \triangle c) + (b \triangle d)$, $(a \setminus b) \triangle (c \setminus d) \leqq (a \triangle c) + (b \triangle d)$, $a \leqq (a \triangle b) + b$ for every a, b, c, $d \in H$; if $a \leqq b$, then $a + c \leqq b + c$, $a \triangle b = b \setminus a$, $a = b \setminus (b \setminus a)$; if $a_n \nearrow a$, $b_n \nearrow b$, $c_n \searrow c$, then $a_n + b_n \nearrow a + b$, $a_n \setminus b \nearrow a \setminus b$, $b \setminus c_n \nearrow b \setminus c$. J_o is assumed moreover to satisfy the following properties: $J_o(0) = 0$, $J_o(a + b) \leqq J_o(a) + J_o(b)$, $J_o(b) = J_o(a \wedge b) + J_o(b \setminus a)$.

If we now put $d(x,y) = J^*(x \triangle y)$ and $H_1 = \{x ; J^*(x) < \infty\}$, then (H_1,d) is a pseudometric space containing A .

3. Finally we put $S = A^-$ (the closure of A with respect to d) and $J = J^* | A^-$.

4. Theorem. S is a sublattice of H closed under + and J is an extension of J_o satisfying the following conditions:

1. If $x \leqq y$, x, $y \in S$, then $J(x) \leqq J(y)$.
2. $J(x) + J(y) = J(x \vee y) + J(x \wedge y)$ for every x, $y \in S$.
3. If $x_n \in S$ (n=1,2,...), $x \in H$, $x_n \nearrow x$ $(x_n \searrow x)$ and $(J(x_n))_{n=1}^{\infty}$ is bounded, then $x \in S$ and $J(x_n) \to J(x)$.

The classical measure extension theorem and Radon measure extension theorem follow immediately from Theorem 4. Of course, these two examples are not the only ones.

5. Theorem. Let G be an Abelian lattice ordered group, which is σ-complete (i.e. every non-empty countable bounded subset of G has the supremum and the infimum). Let F be a subgroup of G closed under the lattice operations. Let there to every $x \in G$ exist $a_n \in F$ (n = 1,2,...) such that $x \leqq \bigvee a_n$. Finally let $I_o : F \to R$ be a linear positive operator such that $x_n \nearrow x$, $x_n \in F$ (n = 1,2,...), $x \in F$, implies $I_o(x_n) \to I_o(x)$.

Then there are a subgroup T of G containing F and closed under the lattice operations and a linear positive operator $I : T \to R$ extending I_o and continuous in the following sense: If $x_n \nearrow x$ $(x_n \searrow x)$, $x_n \in T$ (n = 1,2,...), $x \in G$, and $(I(x_n))_{n=1}^{\infty}$ is bounded, then $x \in T$ and $T(x) = \lim I(x_n)$.

Similar results using different constructions have been studied in [1] - [4] . A detailed elucidation of our results including proofs will appear in the journal Mathematica Slovaca.

References

[1] E. M. Alfsen: Order preserving maps and integration processes. Math. Ann. 149 (1963), 419 - 461.

[2] S. Brehmer: Verbandtheoretische Charakterisierung des Mass- und Integralbegriffs von Carathéodory. Potsdam. Forsch. 1974, 88 - 91.

[3] B. Riečan: On a continuous extension of monotone functionals of some type. (Russian.) Mat.-fyz. časop. 15 (1965), 116 - 125.

[4] M. Šabo: On an extension of finite functionals by the transfinite induction. Math. Slovaca 26 (1976), 193 - 200.