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On a problem of Katětov


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In his 1958 paper [1], M. Katětov studied extensions of locally finite covers. Consider the following statements.

1. $X$ is collectionwise normal and countably paracompact.
2. Every locally finite open family in a closed subspace of $X$ can be extended to a locally finite open family in $X$.
3. Every locally finite functionally open family in a closed subspace of $X$ can be extended to a locally finite open family in $X$.
4. $X$ is collectionwise normal.

In [K], Katětov proved (1) $\rightarrow$ (2) $\rightarrow$ (4) and asked if the converses of these implications hold. Przymusinski recently answered one of Katětov's questions by showing (2) $\leftrightarrow$ (1) under $V = L$. He also noticed (2) $\rightarrow$ (3) $\rightarrow$ (4) and conjectured that (4) $\leftrightarrow$ (3) $\leftrightarrow$ (2). In this paper we settle the remaining question of Katětov by showing that (4) $\leftrightarrow$ (3) $\leftrightarrow$ (2). No set theoretic assumptions beyond the axiom of choice are needed, but we do show how extra assumptions can be used to strengthen the results. Details of results not proved here will appear in [3], along with related results of Przymusinski and the author. Included in [3] will be Przymusinski's work relating Katětov's properties to partitions of unity.

Definitions: If $\{V_{\alpha}\mid \alpha \in \kappa\}$ is a family of subsets of $K$, a family $\{U_{\alpha}\mid \alpha \in \kappa\}$ is said to extend $\{V_{\alpha}\mid \alpha \in \kappa\}$ if $U_{\alpha} \cap K = V_{\alpha}$ for all $\alpha \in \kappa$. A set is called functionally open (= cozero) in $X$ if it can be represented as $\{x \in X\mid f(x) \neq 0\}$ for some continuous function $f : X \rightarrow R$. In a normal space, the functionally open sets are just the open $F_\sigma$ sets. Przymusinski has defined a normal space $X$ to be Katětov if it satisfies (2) above, and to be functionally Katětov if it satisfies (3) above. We will use Przymusinski's terminology throughout the rest of this paper.

§1: Functionally Katětov, not Katětov Spaces.

In this section we give two examples of spaces that are functionally Katětov but not Katětov.

Theorem 1: The Dowker space constructed by M. E. Rudin in [4] is functionally Katětov but not Katětov.

Theorem 2: $V = L$ implies there is a space that is functionally Katětov, locally countable, and of cardinality $\omega_1$, but is not functionally Katětov.

We give Theorem 2 for two reasons. First, the construction for Theorem 2 is easier to work with than Rudin's construction (at least for those readers acquainted with the important technique used by Ostazewski in [2]). Second, $V = L$ allows us...
to build many nice properties into the example. Not only does it have the properties listed above, but (as in [2]) it can easily be modified to be locally compact, hereditarily separable, and first countable. We will prove Theorem 2 in detail since the proof will not be given in [3].

A Sketch of the proof of Theorem 1. We refer the reader to [4] for the construction of Rudin's example, $X$. To show that $X$ is not Katětov, let

$$K = \{ f \in X | f(n) = \omega_n \text{ for some } n \} \text{ and } V_n = \{ f \in X | f(i) = \omega_1 \text{ iff } i = n \}$$

for each $n \in \omega$. Then $K$ is closed in $X$ and $\{ V_n | n \in \omega \}$ is a locally finite open family in $K$ that can be shown to have no locally finite open extension to $X$. The key to proving that $X$ is functionally Katětov is the fact that every $F_\sigma$ in $X$ is closed.

Proof of Theorem 2: Unless otherwise stated, $\alpha$ and $\beta$ will denote countable ordinals, $\lambda$, $\xi$, $\gamma$, and $\eta$ will denote limit ordinals less than or equal to $\omega_1$, and $n$, $m$ and $k$ will denote non-negative integers. Thus a phrase such as "for all $\alpha, \lambda \in A$" should be read "for all $\alpha, \lambda \in A$ such that $\alpha < \omega_1$, $\lambda \leq \omega_1$, and $\lambda$ is a limit ordinal".

Let \( \{ S_\lambda | \lambda < \omega_1 \} \) satisfy

1. for all $\lambda$, $S_\lambda$ is a cofinal subset of $\lambda$ that is cofinal in no smaller limit ordinal, and
2. if $S$ is an uncountable subset of $\omega_1$, then there is a $\lambda$ with $S_\lambda \subseteq S$.

Recall that $V = L$ implies that such a sequence exists.

The construction. Let $Y_\lambda = \omega \times \lambda$ for all $\lambda$. Let $\tau_\omega$ be the discrete topology on $Y_\omega$. For each $\lambda$, inductively construct $\tau_\lambda$ from $\{ \tau_\xi | \xi < \lambda \}$ in the following way.

If $\lambda$ is a limit of limit ordinals, let $\tau_\lambda$ be the topology generated by

$$U(\{ \tau_\xi | \xi < \lambda \})$$

If $\lambda = \xi + \omega$ for some $\xi$, partition $S_\xi$ into $\omega \times \omega$ infinite pieces, $S_{n,m}$. Let $\tau_\lambda$ be the topology generated by all sets, $U$, of any of the following forms:

a) $U \in \tau_\xi$

b) $U = \{(0,\xi + m)\}$ for any $m$.

c) $U = (n,\xi + m) \cup V$ for any $m$ and $V \in \tau_\xi$ such that either

(i) $n$ is odd and $V$ contains all but finitely many points of $\{0\} \times S_{n,m}$,

(ii) $n$ is positive and even and $V$ contains all but finitely many points of $\{n-2, n-1, n\} \times S_{n,m}$.

Let $Y = Y_{\omega_1}$ and $\tau = \tau_{\omega_1}$. Then $(Y, \tau)$ is a functionally Katětov space that is not Katětov.
Verification of the properties of Y. For notational convenience, let $F = \{n\} \times \omega_1$ and set $\pi(A) = \{\alpha | (n, \alpha) \in A$ for some $n\}$ for each $A \subset Y$. Proofs of the first three facts below are left to the reader.

1) $n \times \alpha \in \tau_\lambda$ for all $n, \alpha$ and $\lambda$ with $\alpha < \lambda$.

2) $F_0 \cup F_n \in \tau$ for each odd $n$.

3) $(n, \alpha) \in c\ell_{\xi}^\tau (\{m\} \times \eta)$ whenever $\eta \leq \alpha < \xi$ and either
   a) $0 < n$ is even and $m = n, n - 1$ or $n - 2$, or
   b) $n$ is odd and $m = 0$.

4) If for some $\xi$, $V$ is clopen in $\tau_\xi$ and $\sup \pi V < \xi$, then $V$ is clopen in $\tau_\lambda$ for each $\lambda > \xi$.

Proof: Let $V$ and $\xi$ be as in the hypothesis. For all $\lambda > \xi$, $\tau_\xi \subset \tau_\lambda$ and hence $V$ is open in $\tau_\lambda$. We prove by induction that $V$ is also closed with respect to $\tau_\lambda$. Fix $\lambda > \xi$ and suppose that $V$ is clopen in $\tau_\eta$ for each $\eta$ such that $\xi \leq \eta < \lambda$.

If $\lambda$ is a limit of limit ordinals, then $\bigcup_{\eta < \lambda} \tau_\eta$ is a base for $\tau_\lambda$. Since $X_\eta - V \in \tau_\eta$ for all $\eta < \lambda$, $(X_\lambda - V) = \bigcup_{\eta < \lambda} (X_\eta - V) \in \tau_\lambda$ also.

If $\lambda = \eta + \omega$ for some $\eta$, notice that since $\sup \tau < \eta$, $\pi V$ contains only finitely many points of $S_\eta$. Since $V$ is closed in $\tau_\eta$, $(Y - V) \cup \{(n, \eta + m) \mid \eta < \xi\}$ for all $n$ and $m$. It follows that $Y_\lambda - V \in \tau_\lambda$.

5) $Y$ is regular.

Proof: We prove each $\tau_\lambda$ is regular by induction on $\lambda$. Note that $\tau_\omega$ is regular. Fix $\lambda$ and assume that for each $\xi < \lambda$, $\tau_\xi$ is a regular topology. Let $U \in \tau_\lambda$ and $x \in U$. We will show there is a clopen $V$ with $x \in V \subset U$, and hence $\tau_\lambda$ is regular. There are two cases.

If $\lambda$ is a limit of limit ordinals, then $x \in Y_\xi$ for some $\xi < \lambda$. Since $\tau_\xi$ is regular and $Y_\xi$ is countable, there exists a clopen $V \in \tau_\xi$ with $x \in V \subset U$ and $\sup \pi V < \xi$. Then by (4) $V$ is clopen in $\tau_\lambda$ also.

Now assume $\lambda = \xi + \omega$ for some $\xi$. Since $\tau_\xi$ satisfies (1), the set $\omega \times S_\xi$ is closed and discrete in $X_\xi$. Since $X_\xi$ is countable and regular, there exists a closed discrete collection $\{V_{n, s} \mid n \in \omega, s \in S_\xi\}$ such that $(n, s) \in V_{n, s}$, each $V_{n, s}$ is clopen, and $V_{n, s} \cap V_{m, t} = \emptyset$ unless $n = m$ and $s = t$. In the light of the collection $\{V_{n, s} \mid n \in \omega, s \in S_\xi\}$, the regularity of $\tau_\lambda$ easily follows from the regularity of $\tau_\xi$ and the definition of $\tau_\lambda$.

6) Any two closed uncountable subsets of $Y$ intersect.

Proof: Let $H$ and $K$ be uncountable closed subsets of $Y$. Then for some $n$, $H$ has an uncountable intersection with $\{n\} \times \omega_1$ and hence contains $\{n\} \times S_\gamma$ for some $\gamma$. By (3-a), this implies that $H$ contains all but countably many points of $F_k$, where $k$ is the first even integer greater than $n$. By applying (3-a) repeatedly, it follows that $H$ contains all but countably many points of $F_j$
for each even \( j > k \). Since a similar statement holds for \( K \), we have \( H \cap K \neq \emptyset \).

7) \( Y \) is normal.

Proof: Let \( H \) and \( K \) be two disjoint closed subsets of \( Y \). By (6), at least one of \( H \) and \( K \) is countable. Without loss of generality we assume \( H \) is countable. Choose \( \lambda < \omega \) such that \( \lambda > \sup \pi H \). Then since \( Y_\lambda \) is countable and regular, there is a \( V \), clopen in \( \tau_\lambda \), such that \( H \subseteq V \) and \( V \cap K = \emptyset \). Moreover, by (1), \( V \) can be chosen so that \( \sup \pi V < \lambda \). Then (4) implies that \( V \) is a clopen separation of \( H \) and \( K \) in \( Y \). \( \square \)

8) \( Y \) is functionally Šaksčov.

Proof: Assume \( K \) is closed in \( Y \) and \( U \) is a locally finite functionally open family of \( K \). We must show that \( U \) has a locally finite open extension to \( Y \). First note that \( U \) must be countable, for is not, choose \( x_\lambda \in U \) for each \( U \in U \).

Since \( U \) is locally finite, \( \{ x_\lambda \mid U \in U \} \) is an uncountable closed discrete collection in \( Y \). But every uncountable subset of \( Y \) contains \( \{ \} \times S_\lambda \) for some \( n \) and \( \lambda \), and hence is not closed discrete. Write \( U \) as \( \{ U_n \mid n \in \omega \} \).

Next observe that if any \( U_n \) is uncountable, then since it is functionally open, it contains an uncountable closed set, and hence \( K \cup U_n \) is countable (by arguments similar to those used in (3-a) and (6)). Thus only finitely many of the \( U_n \) are uncountable and we can assume without loss of generality that none of the \( U_n \) are uncountable.

Choose \( \xi > \sup (U_n) \) for all \( n \) with \( \xi \) so large that each \( \{ \} \times (\omega_1 - \xi) \) has a neighborhood, \( W_n \), in \( Y \) whose closure intersects only finitely many \( U_n \).

(The reader can check that such an \( \xi \) actually exists.) \( Y_\xi \) is regular and countable, and hence Šaksčov, so there exists a locally finite (in \( Y_\xi \)) open collection \( \{ V_n \mid n \in \omega \} \) such that \( V_n \cap K = U_n \).

Let \( V_n' = V_n \cup \{ \} \times m \mid m < n \) and \( \tilde{W}_n \cap U_n = \emptyset \).

Then \( \{ V_n' \mid n \in \omega \} \) is the desired extension of \( \{ U_n \mid n \in \omega \} \). \( \square \)

9) \( Y \) is not Šaksčov.

Proof: Let \( K = Y - F_0 \) and \( U = \{ F_n \mid n \text{ is odd} \} \). By (1) and (2), \( U \) is a locally finite open family in the closed subspace \( K \). If \( U \) is open in \( Y \) and \( U \cap K = F_n \) for some odd \( n \), then (3-b) implies that \( F_0 \cap U \) is countable. It follows that the family \( U \) has no locally finite open extension to \( Y \), and hence \( Y \) is not Šaksčov. \( \square \)

§2: Collectionwise Normal, not Functionally Šaksčov Spaces.

In this section we modify Rudin's example, \( X \), to produce an example, \( Z \), of a space that is collectionwise normal but not functionally Šaksčov. The modification can be done on any "nice" Dowker space to produce an example. Using \( V = L \), one can construct a collectionwise normal, not functionally Šaksčov space that is
hereditarily separable, locally countable, and of cardinality \( \omega_1 \).

Let \( W_n = \{ f \in X | f(i) = \omega_1 \text{ when } i \leq n \text{ and } \text{cf}(f(1)) \leq \omega_n \text{ for all } i > n \} \) for each \( n \geq 1 \). Set \( W = \bigcup\{W_n | n \geq 1\} \) and give \( W \) the subspace topology from \( X \).

For \( n,m \in \omega \) and \( A \subset W \), let \( A^{n,m} \) denote \( A \times (n,m) \) and \( A^n \) denote \( A \times \{n\} \). Define

\[
Z = W \cup \{W_1^{n,m} | n, m \in \omega, m > 1\} \cup \{W_k^{m} | k > 1, m \in \omega\}.
\]

We generate a base for \( Z \) from the open sets in \( W \). For each \( U \), open in \( W \), and each \( n,m \in \omega \) with \( m > 1 \) and sequence of integers \( \{k_i | i \in \omega\} \), the following two sets are declared to be basic open subsets of \( Z \):

\[
(U \cap W_1)^{n,m} \cup (U \cap W_m)^n \\
\cup \cup \{U \cap W_1^{i,j} | i > k_j \} \cup \cup \{U \cap W_j^{i,j} | i > k_j\}.
\]

Then \( Z \) is a collectionwise normal space that is not functionally Katětov. We will sketch the proof that \( Z \) has the desired properties; details will appear in [3].

To see that \( Z \) is not functionally Katětov, let

\[
K = (W - W_1) \cup \{W_m^{n} | n, m \in \omega, m > 1\} \text{ and set } V_m = \{W_m^n | n \in \omega\} \text{ for each } m > 1.
\]

Then \( \{V_m | m > 1\} \) is a locally finite functionally open family in the closed subspace \( K \) that can be shown to have no locally finite open extension to \( Z \).

The proof that \( Z \) is collectionwise normal is the hard part of this example.

The following difficult fact is used repeatedly in the proof.

**Lemma:** For each \( n \), \( W_1 \cup W_n \) is collectionwise normal and countably paracompact.

§3: **Katětov Spaces that are not Countably Paracompact.**

We have already mentioned that Przymusiński has used \( V = L \) to construct a Katětov space that is not countably paracompact. He does this by constructing a hereditarily normal, hereditarily separable Dowker space, and then showing that every such space is Katětov. It would be nice to have an example of a Katětov space that is not countably paracompact whose construction does not use set theoretic assumptions beyond the axiom of choice. The author conjectures that the space \( W \) (constructed in §2) is such a space. It is hoped that this conjecture will be settled in [3].

**REFERENCES**


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