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RELATIVE COMPACTNESS AND RECENT COMMON GENERALIZATIONS OF METRIC AND LOCALLY COMPACT SPACES

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Introduction

By assumption of a "good" connection between compact subsets and the topology of a space, a number of new classes of topological spaces have been introduced and investigated in the last fifteen years. Spaces obtained in this way are p-spaces of Argangel'skiĭ (see [7]), paracompact p-spaces (or paracompact M-spaces) invented independently by the Moscow and Japan school of point set topology, spaces of first-countable type (see [2], e.g.), spaces with small K-bases (see [13]), and others.

It seems that most of the results obtained for the above-mentioned classes can be extended and unified by using a simple and natural idea, which is relative compactness. We say that a topology τ defined in a non-void set X is compact relative to a topology τ' in X if each ultrafilter which is convergent in τ' is also convergent in τ.

The aim of this note is to make a brief summary of our results concerning relative compactness. § 1 contains introductory results. §2 and §3 are devoted to topics which are motivated by important results concerning the classes mentioned in the first sentences of our introduction. In §4 we solve a problem of Arhangel'skiĭ (affirmatively, under CH), by proving even a more general result concerning relative compactness. Finally, in §5 we obtain from the results of §2-§4 a number of known and unknown results as corollaries for p-spaces, K-bases etc.

Remarks on terminology and notation. Throughout the paper $\varkappa$ denotes an infinite cardinal. Regular spaces are supposed to be T_1 spaces. The use of the generalized continuum hypothesis will be denoted by GCH. Given a topological space (X,τ), and a subset A of X, $cl_\tau A$ denotes the closure of A in τ. If X is fixed then we shall sometimes say that τ has a certain property instead of telling that (X,τ) has that property.

1§. Some properties and examples of relative compactness

First we recall some definitions. The Lindelöf degree of a topological space (X,τ) is the smallest infinite cardinal $\varkappa$ such that every τ-open cover of X contains a subcover of cardinality $\leq \aleph$. The <u>paracompactness degree</u> of (X,τ) is the smallest infinite cardinal $\aleph$ such that each τ-open cover of X has a τ-open refinement which is the union of at most $\aleph$ locally finite families of (X,τ). (See [14].) Note that a regular space is paracompact if and only if its paracompactness degree is ω. (See [11].) A collection $\{\mathcal{U}_i : i \in I\}$ of open covers of a topological space (X,τ) is called a <u>pluming</u> for (X,τ) if the following holds: if $x \in G_i \in \mathcal{U}_i$ for all i in I then

(a) $C_x = \bigcap\{cl_\tau G_i : i \in I\}$ is compact in τ;

(b) $\{\bigcap\{cl_\tau G_i : i \in J\} : J$ is a finite subset of $I\}$

is a "base" for C_x, i.e. given any open subset U containing C_x, there is a finite subset J of I with $\bigcap\{cl_\tau G_i : i \in J\} \subset U$. (See [9].) In [7] it is shown that a regular space is a <u>p-space</u> in the sense of Arhangel'skiĭ [1] if and only if it has a pluming $\{\mathcal{U}_i : i \in I\}$ with $|I| \leq \omega$. A cover $\mathcal{L}$ of a space (X,τ) is called a <u>K-net</u> for (X,τ) if for each point x in X, there exists a compact subset C_x of (X,τ) such that for every τ-neighbourhood V of C_x, there is a member B in $\mathcal{L}$ with $x \in B \subset V$. A τ-open K-net is called a <u>K-basis</u> for (X,τ). (See [13].)

<u>Lemma 1.1.</u> Let τ and τ' be two topologies defined in the same non-void set X. Then the following conditions are equivalent.

(i) Every ultrafilter, which is convergent in τ', also converges in τ (i.e. τ is compact relative to τ').

(ii) Every filter-base, which has a cluster point in τ', has a cluster point in τ, too.

(iii) For every τ-open cover $\mathcal{U}$ of X, and for every point x in X, there is a τ'-neighbourhood of x which is covered by a finite subfamily of $\mathcal{U}$.

<u>Theorem 1.2.</u> Let $\{(X_i,\tau_i) : i \in I\}$ and $\{(X_i,\tau_i') : i \in I\}$ be families of topological spaces such that τ_i is compact relative to τ_i' for all i in I. Then the topology of the product $\times\{(X_i,\tau_i) : i \in I\}$ is compact relative to the topology of $\times\{(X_i,\tau_i') : i \in I\}$.

<u>Theorem 1.3.</u> Suppose that the topology of a space (X,τ) is compact relative to a topology τ' in X. Then the Lindelöf degree of τ does not exceed that of τ'. Moreover, if τ' is weaker than τ then the paracompactness degree of τ does not exceed that of τ'.

<u>Proposition 1.4.</u> Let $\mathcal{L}$ be a K-net for a topological space (X,τ), and let τ' be the topology in X which is generated by $\mathcal{L}$ as

a subbase. Then τ is compact relative to τ'.

Proposition 1.5. Let $\{\mathcal{U}_i : i \in I\}$ be a pluming for a topological space (X,τ), and let τ' be the topology in X which is generated by $\mathcal{U} = \bigcup\{\mathcal{U}_i : i \in I\}$ as a subbase. Then τ is compact relative to τ'.

2§. Extension of a theorem of J.Nagata to relative compactness

We beginn with some definitions. The metrizability degree of a topological space (X,τ) is the smallest infinite cardinal $\varkappa$ such that (X,τ) has an (open) base which is the union of at most $\varkappa$ locally finite families of (X,τ). By virtue of the classical Nagata - Smirnov metrization theorem regular spaces with metrizability degree ω are exactly the metrizable spaces. (For a discussion of the metrizability degree see Hodel's paper [11].) Let X be non-void set, let $\mathcal{U}$ be a family of subsets of X. Let us define the pointwise cardinality of $\mathcal{U}$ as the smallest infinite cardinal $\varkappa$ such that every element of X is contained in at most $\varkappa$ members of $\mathcal{U}$. $\mathcal{U}$ is said to be point-countable if it has pointwise cardinality ω. Finally, let us call a cover $\mathcal{U}$ of a topological space (X,τ) separating (resp. strongly separating) if for every pair of distinct points x, y in X, there is a G in $\mathcal{U}$ with $x \in G, y \notin G$ (resp. with $x \in G, y \notin cl_\tau G$).

The aim of this paragraph is to extend the following theorem of J.Nagata to relative compactness (in Theorem 2.2): Every paracompact p-space with a point-countable separating open cover is metrizable. (See [16]. For an interesting story of how this result was developed step by step by several authors see R.E.Hodel [11] who has extended it to higher cardinality.) Note, further, that paracompact p-spaces (or paracompact M-spaces) are exactly the spaces having a perfect map onto a metrizable space. (See [1].)

Lemma 2.1. Let us suppose that the topology of a space (X,τ) is compact relative to a weaker topology τ' in X with metrizability degree $\leq \varkappa$, and that there is a separating τ-open cover (resp. a strongly separating τ-open cover, resp. a base) for (X,τ) of pointwise cardinality $\leq \varkappa$. Then there is a separating τ-open cover (resp. a strongly separating τ-open cover, resp. a base) for (X,τ) which is $\varkappa$-locally finite in τ.

Theorem 2.2. If the topology of a regular space (X,τ) is compact relative to a weaker topology τ' in X with metrizability degree

$\leq \varkappa$, and (X,τ) has a separating open cover of pointwise cardinality $\leq \varkappa$ then (X,τ) has metrizability degree $\leq \varkappa$.

For regular spaces, the following result is a corollary of Theorem 2.2.

Theorem 2.3. If the topology of a Hausdorff space (X,τ) is compact relative to a weaker topology τ' in X of weight $\leq \varkappa$, and (X,τ) has a separating open cover of pointwise cardinality $\leq \varkappa$ then (X,τ) has weight $\leq \varkappa$.

3.§. Concerning the preservation of the tightness, character and weight of topologies under hereditary assumptions

Recall that the tightness of a point x in a topological space (X,τ), denoted $t(x,X,\tau)$, is defined to be the smallest infinite cardinal $\varkappa$ such that for every subset A of X with $x \in cl_\tau A$, there is a subset A_1 of A with $|A_1| \leq \varkappa$ and $x \in cl_\tau A_1$. The pseudocharacter of x in (X,τ), denoted $\psi(x,X,\tau)$, is the smallest infinite cardinal $\varkappa$ such that there is a family $\mathcal{W}$ of τ-open subsets with $|\mathcal{W}| \leq \varkappa$ and $\bigcap \mathcal{W} = \{x\}$. Denote by $\chi(x,X,\tau)$ the character of the point x in (X,τ). Denote by $\chi(X,\tau)$ the character of (X,τ), i.e. $\chi(X,\tau) = \sup\{\chi(x,X,\tau) : x \in X\}$. $t(X,\tau)$ and $\psi(X,\tau)$ can be defined similarly.

The two main results in this paragraph are Theorem 3.2 and Theorem 3.5.

Lemma 3.1. If the topology of a regular space (X,τ) is compact relative to a weaker topology τ' in X then

$$t(x,X,\tau) \leq \psi(x,X,\tau) \cdot t(x,X,\tau') \quad \text{and}$$
$$\chi(x,X,\tau) \leq \psi(x,X,\tau) \cdot \chi(x,X,\tau')$$

for every point x in X.

Theorem 3.2. Let (X,τ) be a regular space, let x be a point in X. Suppose that the topology of every subspace (Y,τ_Y) containing x is compact relative to a weaker topology τ'_Y in Y with $t(x,Y,\tau'_Y) \leq \varkappa$. Then $t(x,X,\tau) \leq \varkappa$.

Corollary 3.3. If the topology of each subspace (Y,τ_Y) of a regular space (X,τ) is compact relative to a weaker topology τ'_Y in Y with $t(Y,\tau'_Y) \leq \varkappa$ then (X,τ) has tightness $\leq \varkappa$.

If the topology of each subspace of a regular space (X,τ) is compact relative to a weaker first countable topology then (X,τ) need not be first countable. (A suitable counter-example is the one-point compactification of any uncountable discrete space.) However, the following result can be deduced from Theorem 3.2.

<u>Theorem 3.4.(GCH).</u> Suppose that the topology of each subspace (Y, τ_Y) of a regular space (X, τ) is compact relative to a weaker topology τ'_Y in Y with $\chi(Y, \tau'_Y) \leq \kappa$. Then there is a τ-open subset Y^* of (X, τ) such that Y^* is dense in (X, τ) and $\chi(y, X, \tau) \leq \kappa$ for each y in Y^*.

<u>Theorem 3.5.</u> Suppose that $\kappa = \omega$ or $2^\kappa = \kappa^+$. Then if the topology of every subspace (Y, τ_Y) of a regular space (X, τ) is compact relative to a weaker topology τ'_Y in Y of weight $\leq \kappa$ then (X, τ) has weight $\leq \kappa$.

4§. <u>On a problem of Arhangel'skiǐ</u>

It was posed in Arhangel'skiǐ [4] wether a space, each subspace of which is a paracompact p-space, contains a dense metrizable subspace. We have solved this problem affirmatively, if CH holds (Corollary 5.7.1). However, a more general theorem is valid, which is announced in this paragraph as Theorem 4.2.

<u>Lemma 4.1.</u> Let (X, τ) be a regular space with character $\leq \kappa$ and metrizability degree $\leq \kappa^+$. Suppose that the topology of every subspace (Y, τ_Y) of (X, τ) is compact relative to a weaker topology τ'_Y in Y with metrizability degree $\leq \kappa$. Then (X, τ) contains a dense subspace with metrizability degree $\leq \kappa$.

<u>Theorem 4.2. (GCH).</u> Suppose that the topology of every subspace (Y, τ_Y) of a regular space (X, τ) is compact relative to a weaker topology τ'_Y in Y with metrizability degree $\leq \kappa$. Then (X, τ) contains a dense subspace with metrizability degree $\leq \kappa$.

5§. <u>Corollaries</u>

Denote the weight, Lindelöf degree, paracompactness degree, and metrizability degree of a topological space (X, τ) by $w(X, \tau)$, $L(X, \tau)$ $pa(X, \tau)$ and $m(X, \tau)$, respectively. Recall that the <u>Souslin number</u> of (X, τ), denoted $c(X, \tau)$, is the smallest infinite cardinal κ such that every family of pairwise disjoint τ-open subsets has cardinality $\leq \kappa$. The <u>point separating weight</u> of (X, τ), denoted $psw(X, \tau)$, is defined to be the smallest infinite cardinal κ such that (X, τ) has a separating open cover of pointwise cardinality $\leq \kappa$. (See [9].)

The plurning degree of a regular space, denoted $pl(X,\tau)$, is the smallest infinite cardinal κ such that (X,τ) has a pluming $\{\mathcal{U}_i : i \in I\}$ with $|I| \leq \kappa$. (See [9], too.) The K-weight of (X,τ), denoted $Kw(X,\tau)$, is the smallest infinite cardinal κ such that (X,τ) has a K-basis of cardinality $\leq \kappa$. (See [13].) Let us say that (X,τ) is of point-κ type if for every point x of X, there is a subset C of X containing x such that C is compact in τ and has character $\leq \kappa$ in τ. Spaces of point-ω type are called spaces of point-countable type. Finally, if $f(X,\tau)$ is a cardinal invariant then write $f^*(X,\tau) = \sup\{f(Y,\tau_Y) : (Y,\tau_Y) \text{ is a subspace of } (X,\tau)\}$.

In order to obtain corollaries to our results, Proposition 1.4 and 1.5 are useful.

Corollary 5.1. If a regular space (X,τ) has a K-basis which is the union of at most κ locally finite families of (X,τ), and $psw(X,\tau) \leq \kappa$ then (X,τ) has metrizability degree $\leq \kappa$.

Corollary 5.2. (Hodel [11].) If (X,τ) is a regular space then

$$m(X,\tau) = pl(X,\tau) \cdot pa(X,\tau) \cdot psw(X,\tau).$$

Both of the above corollaries follow from Theorem 2.2.

Corollary 5.2.1. (Hodel [9].) If (X,τ) is a regular space then

$$w(X,\tau) = pl(X,\tau) \cdot L(X,\tau) \cdot psw(X,\tau).$$

Corollary 5.2.2. (Nagata [16].) A paracompact p-space with a point-countable separating open cover is metrizable.

Corollary 5.3. (GCH) If each subspace of a regular space (X,τ) is of point-κ type then there is a τ-open subset Y of X such that Y is dense in (X,τ) and $\chi(y, X, \tau) \leq \kappa$ for each y in Y.

For $\kappa = \omega$, this result is proved in Arhangel'skiĭ [4]. We deduced this result from our Theorem 3.2.

Corollary 5.4. (Juhász [13].) Suppose that $\kappa = \omega$ or $2^\kappa = \kappa^+$. Then if each subspace of a regular space (X,τ) has K-weight $\leq \kappa$ then (X,τ) has weight $\leq \kappa$.

Corollary 5.5. (Hodel [10].) Let (X,τ) be a regular space. Then $w(X,\tau) = L^*(X,\tau) \cdot pl^*(X,\tau)$ if for $\kappa = L^*(X,\tau) \cdot pl^*(X,\tau)$ either $\kappa = \omega$ or $2^\kappa = \kappa^+$ holds.

The two above corollaries follow from Theorem 3.5.

Corollary 5.5.1. (Hodel [10].) Let (X,τ) be a topological space, and suppose that for $\kappa = c(X,\tau)$ either $\kappa = \omega$ or $2^{\kappa} = \kappa^{+}$ holds. Then if each subspace of (X,τ) is a paracompact p-space then $c(X,\tau) = w(X,\tau)$.

Corollary 5.6. (GCH) If each subspace (Y,τ_Y) of a regular space (X,τ) has a κ-basis which is the union of at most κ locally finite families of (Y,τ_Y) then (X,τ) has a dense subspace with metrizability degree $\leq \kappa$.

Corollary 5.7. (GCH) If for each subspace (Y,τ_Y) of a regular space (X,τ) $pl(Y,\tau_Y)\cdot pa(Y,\tau_Y) \leq \kappa$ holds then contains a dense subspace with metrizability degree $\leq \kappa$.

Corollary 5.6 and 5.7 follow from Theorem 4.2.

Corollary 5.7.2. (CH) Every space, each subspace of which is a paracompact p-space, contains a dense metrizable subspace.

As we have already indicated the last corollary answers Problem 4 in Arhangel'skiǐ [4], affirmatively.

The proofs will appear in [5] and [6].

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