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In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [338]--341.

Persistent URL: http://dml.cz/dmlcz/700698

SEPARATE CONTINUITY AND CONTINUITY FOR SOME GENERALIZED CONTINUITY NOTIONS

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The relation between separate continuity and continuity depends on the type of continuity which is considered. The separate quasicontinuity implies quasicontinuity of a function f as a function of two variables. (See [5] for a real function of two real variables and [7] for a mapping $f : X \times Y \to Z$ where X is a Baire space, Y second-contable and Z regular.) The converse is not true. The feeble continuity [1] (somewhat continuity [2]) in each variable separately of a function f on X x Y does not imply the feeble somewhat continuity of f as a function of twoo variables. But somewhat continuity in one variable and quasicontinuity in the other give some information of f as a function of two variables. The situation is completely different in case of so called almost continuity [3] , [4] . **The notion of almost oontinuity appears first in [11] .**

<u>Definition 1.</u> If X, Y are topological spaces then a function $f : X \to Y$ is said to be quasicontinuous at $x_o \in X$ if for any open containing x_o, and any open V containing $f(x_o)$, there exists a non-empty open set G such that $f(G) \subset V$. The function f is said to be quasicontinuous, if it is quasicontinuous at any $x \in X$.

<u>Definition 2.</u> A function $f : X \to Y$ is said to be somewhat continuous if for any open $G \subset Y$ such that $f^{-1}(G) \neq 0$, int $f^{-1}(G) \neq \emptyset$ holds.

<u>Definition 3.</u> A function $f : X \to Y$ is said to be almost continuous at $x_o \in X$ if for any V open $V \subset Y$, containing $f(x_o)$, the set $Cl(f^{-1}(V))$ contains a neighbourhood of x_o. We say that f is almost continuous if it is almost continuous at any $x \in X$.

<u>Theorem 1.</u> Let X be a Baire space, Y such that each point $y \in Y$ posesses a neighbourhood satisfying second contability axiom and Z a regular space. Let $f : X \times Y \to Z$ be such that f^y quasicontinuous for each $y \in Y$ and the x - sections f_x, are quasicontinuous with the exception of a set of first cathegory. Then f is quasicontinuous

Theorem 2. Let X be a Baire space, Y a space satisfying second countability axiom and Z a regular space. Let $f : X \times Y \to Z$ be such that for each $y \in Y$ the sections f^y are quasicontinuous and the x-sections f_x, with the exception of a set of first category are somewhat continuous. Then f is somewhat continuous.

Theorem 1 is evidently a slight generalization of Martin's version of Kempisty's theorem (see [5] [7]) Theorem 2 seems to be of a different kind but it includes (see [8])the mentined Martin's theorem.

It seems to be an interesting fact that in Theorem 2, which resembles to a certain extent Theorem 1, we can not substitute the assumption of second countability by a "locally" second countability as it is in Theorem 1.

Example 1. $T = \langle 0,1)$ will serve as an index set. To each $t \in T$ an isometric image of the metric space $X = \langle 0,1)$ (with the usual metric) will be associated. We may suppose $Y_t \cap Y_{t'} = \emptyset$ for $t \neq t'$. If necessary we shall denote y_t the corresponding image of $y \in \langle 0,1)$ in the space Y_t. If there will be no confusion possible, we write simply y instead of y_t. The sets Y_t are supposed to be endowed with the order structure inherited from $\langle 0,1)$ Put $Y = \bigcup_{t \in T} Y_t$. As to the topology, G is open in Y if $G = \bigcup_{t \in T} G_t$ where G_t are open in Y_t. $R = (-\infty, \infty)$ is considered with the usual topology. We can see that in Y any point y posesses a neighbourhood satisfying second countability axiom. For any $t \in T$ the function

$$^{(t)}f : X \times Y_t \to R \text{ is defined as:}$$

$$^{(t)}f(x,y) = \begin{cases} 0, \text{if } x < t,\ y\ (= y_t),\ \text{rational} \\ 1, \text{if } x < t,\ y \text{ irrational} \\ 0, \text{if } x = t,\ 0 \leq y \leq \frac{1}{2} \\ 1, \text{if } x = t,\ \frac{1}{2} < y < 1 \\ 0, \text{if } x > t,\ y \text{ irrational} \\ 1, \text{if } x > t,\ y \text{ rational} \end{cases}$$

On the product $X \times Y$ define $f : X \times Y \to R$ as:

$$f(x,y) = {}^{(t)}f(x,y), \text{ if } y \in Y_t$$

For any $y \in Y$, f^y is a quasicontinuous function.

For any $x \in X$ the function f_x is somewhat continuous.

The function f is not somewhat continuous as a function of two variables. In fact, if $G = (\frac{1}{2}, \frac{3}{2})$, then $f^{-1}(G) \neq \emptyset$, but $\operatorname{int} f^{-1}(G) = \emptyset$.

By means of a similar example it may be shown that the assumption on X to be a Baire space in Theorem 2 is essential.

In theorems 1 and 2 the quasicontinuity may be substituted by semicontinuity as defined by Levine [6]. It follows from the fact that the last two notions are equivalent as was proved in [10].

As to the almost continuity, there is no good relation between separate almost continuity and almost continuity.

Theorem 3. Let X, Y be separable metric spaces without isolated points. Then there exists a real function $f : X \times Y \to R$, such that f is almost continuous at each $(x, y) \in X \times Y$, and a dense set $D \subset X \times Y$ such that for each $(x_0, y_0) \in D$ the sections f_{x_0} and f^{y_0} are not almost continuous.

The fact that almost continuity of sections does not imply the almost continuity of f as a function of two variables may be also easily **verified.**

Example 2. On the interval $\langle -1,1 \rangle \times \langle -1,1 \rangle$ consider the set $F = \left\{ (x, y) : 0 \leqq x \leqq 1, \frac{1}{2} x \leqq y \leqq x \right\}$

Define

$f : \langle -1,1 \rangle \times \langle -1,1 \rangle \to R$, as

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) \in F - (0,0) \\ 0, & \text{if both } x, y \text{ are simultaneously rational or irrational and } (x,y) \notin F \\ 1, & \text{if } x \text{ is rational, } y \text{ irrational or conversly and } (x,y) \notin F \end{cases}$$

$f(0,0) = 1.$

The function f is not almost continuous at $(0,0)$. The almost continuity of the sections f_{x_0}, f^{y_0} may be easily verified for each $x_0 \in X$, $y_0 \in Y$ respectively.

Detailed proofs will be given in [9].

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