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ON A CLASS OF TOPOLOGICAL SPACES CONTAINING ALL BICOMPACT AND ALL CONNECTED SPACES [1)]

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When comparing a class $\mathcal{B}$ of bicompact spaces and a class $\mathcal{C}$ of connected spaces, it is conspicuous that in spite of all their outward dissimilarity they have some common (more precisely-analogous) properties that moreover belong to the basic topological qualities of these classes. For example:

1) A product of topological spaces is bicompact (connected) iff every factor is bicompact (resp. connected).

2) An image of a bicompact (connected) space under a continuous mapping is bicompact (resp. connected).

3) Let $\mathcal{F}\mathcal{B}$ ($\mathcal{F}\mathcal{C}$) be a collection of all closed mappings having the property that the preimage of every point is bicompact (resp. connected). Then a space X which is a preimage of a space $y \in \mathcal{B}$ ($y \in \mathcal{C}$) under a mapping $f \in \mathcal{F}\mathcal{B}$ ($f \in \mathcal{F}\mathcal{C}$) is also bicompact (resp. connected).

The aim of this paper is to define and to begin the study of a class of spaces (we call them cb-spaces or "clustered" spaces), containing all bicompact and all connected spaces, which possesses some properties common for $\mathcal{B}$ and $\mathcal{C}$. For example, a product of topological spaces is a cb-space iff the same holds for every factor, a continuous image of a cb-space is a cb-space. Although the third property does not hold for cb-spaces to the full extent (see Example 7) there is a certain analogy of it (see Theorem 4, and also Theorem 5).

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1) The detailed version of this paper is to be published in Utchenije Zapisky of the Latvian State University Issue "Topological spaces and their mappings" No 4 (1978).

§ 1. DEFINITION AND BASIC PROPERTIES OF cb-SPACES

Definition. A topological space X is called "clustered" or a cb-space if its every cover consisting of clopen sets (i.e. closed and open) has a finite subcover.

Further a cover consisting of clopen sets will be called a "clopen cover".

It is easy to see that bicompact spaces and also connected spaces are clustered (see Examples 1,2).

Taking into consideration the fact that a complement of a clopen set is again a clopen set one can easily prove the following

Theorem 1. A topological space X is a cb-space iff every centered system of its clopen subsets has a non-void intersection.

From Zorn's lemma one obtains that every centered system of clopen sets is contained in a (unique) maximal centered system of clopen sets. Calling maximal centered systems of clopen sets clopen ultrafilters one can get the following corollary from the previous theorem:

Corollary. A topological space X is a cb-space iff every clopen ultrafilter converges in X.

Theorem 2. A clopen subset of a cb-space is again a cb-space.

The proof is obvious.

Theorem 3. An image of a cb-space under a continuous mapping is a cb-space.

The proof can be easily obtained directly from the definition .

Corollary. A quotient of a cb-space is a cb-space.

Theorem 4. Let f be a clopen mapping [1] of a space X onto a cb-space Y. If the preimage of every point $y \in Y$ under f is clustered then X itself is a cb-space.

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1) A mapping $f: X \to Y$ is called clopen if it maps every clopen set onto a clopen set.

Proof: Consider a family $\mathcal{U} = \{U_\alpha\}$ of clopen subsets in the space X having a void intersection. To prove the theorem it is necessary and sufficient to show that $\mathcal{U}$ is not centered. Clearly, without a loss of generality one may consider that $\mathcal{U}$ is closed under finite intersections.

Let us examine the family of clopen sets $f(U_\alpha)$ in the space Y. First we show that $\cap f(U_\alpha) = \emptyset$. Really, if there exists a point y_0 in $\cap f(U_\alpha)$ then $f^{-1}(y_0)$ would have a non-void intersection with every U_α and hence the family $\{f^{-1}(y_0) \cap U_\alpha\}$ is centered. Since $f^{-1}(y_0)$ is a cb-space, the intersection $\bigcap_\alpha (f^{-1}(y_0) \cap U_\alpha)$ is non-void and hence $\cap U_\alpha \neq \emptyset$ which contradicts our conditions. This contradiction implies that $\cap f(U_\alpha) = \emptyset$ and as Y is a cb-space, the family $\{f(U_\alpha)\}$ cannot be centered. Find indexes $\alpha, \ldots, \alpha_n$ such that $\bigcap_i f(U_{\alpha_i}) = \emptyset$, then obviously $\bigcap_i U_{\alpha_i} = \emptyset$ and so the family $\mathcal{U} = \{U_\alpha\}$ is not centered. The theorem is proved.

Using similar ideas it is easy to prove the following proposition:

Theorem 5. Let f be a mapping of a space X onto a bicompact space Y, the preimage of every point $y \in Y$ under f being clustered. Then X is also a cb-space.

Theorem 6. Let X be a completely regular space and βX its Stone-Čech compactification. The space X is clustered iff its every cover consisting of clopen in βX subsets has a finite subcover.

The proof follows from the well-known fact (see e.g. [1]) that the closure of a clopen subset $A \subset X$ in the extension is clopen in it.

Theorem 7. A product of topological spaces is a cb-space iff all factors are cb-spaces.

Proof: If the product $X = \prod X_\alpha$ is a cb-space, then so is every factor X_α, because X_α is the image of X under corresponding projection $\pi_\alpha : X \to X_\alpha$ (see Theorem 3).

Conversely, if every X_α is clustered then the product $X = \prod X_\alpha$ is also a cb-space. This fact may be proved by virtue of a number of auxiliary propositions some of which we suppose to be of interest by themselves.

Lemma 1. Let Y be a cb-space, X -any topological space and $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ - the corresponding projections. If π_Y

is a clopen mapping, then the mapping π_X is also clopen.

Using Lemma 1 it is easy to prove

<u>Lemma 2.</u> If X is bicompact and Y is a cb-space, then the projections π_X and π_Y are clopen mappings.

Lemma 2 implies

<u>Proposition 1.</u> A projection along a cb-space is a clopen mapping.

This proposition in its turn allows us to prove

<u>Proposition 2.</u> The closure of a projection of a clopen subset of the product is a clopen subset in the corresponding factor.

<u>Lemma 3.</u> Let $z, c \in \Pi X_\alpha$, moreover $z \notin W$ nad $c \in W$ where W is a clopen subset in the product. Then there exists a factor X_α and a clopen set $U_\alpha \subset X_\alpha$ such that $z \in \pi_\alpha^{-1}(U_\alpha)$ but $c \notin \pi_\alpha^{-1}(U_\alpha)$.

It is convenient to prove this lemma first for two factors, then to extend it to the finite case and finally to use Proposition 2 to prove the general case.

With the aid of Lemma 3 and Proposition 1 one can prove the following

<u>Lemma 4.</u> Let X, Y be cb-spaces, W- a clopen subset of the product $X \times Y$ and z- a point that does not belong to W. Then there exist clopen sets $U \subset X$, $V \subset Y$ such that $(U \times V) \cap W = \phi$ but $z \in U \times V$.

By Lemma 4 and Proposition 1 we may prove the following

<u>Lemma 5.</u> The product of two cb-spaces is a cb-space.

<u>Corollary.</u> The product of a finite number of cb-spaces is also a cb-space.

Further, using this corollary, Lemma 3 and Theorem 2, it is not difficult to prove

<u>Lemma 6.</u> If $X_1, X_2, \ldots, X_n$ are cb-spaces and W is a clopen

subset of the product $X = \prod X_i$ then there exist clopen sets $U_1 \subset X_1$ $U_2 \subset X_2, \ldots, U_n \subset X_n$ such that $z \in \prod_i U_i$ and $(\prod U_i) \cap W = \phi$.

Now, using Proposition 2, one may generalize Lemma 6 :

Proposition 3. Let X_α be a cb-space for every index $\alpha \in A$, W– a clopen subset in the product $X = \prod X_\alpha$ and $z \notin W$. Then there exists a clopen set $V \subset X$ such that $z \in V$, $V \cap W = \phi$ and, moreover, V is of the form $V = \prod U_\alpha$ where every U_α is a clopen subset of the corresponding factor X_α and $U_\alpha = X_\alpha$ for all but a finite number of indexes α .

Now we pass directly to the proof of Theorem 7. Let X_α be a cb-space and $\mathcal{F}$–a clopen ultrafilter in the product $X = \prod X_\alpha$. For every α consider a family of subsets $\mathcal{F}_\alpha = \{\overline{\pi_\alpha(U)} : U \in \mathcal{F}\}$. Every $\pi_\alpha(U)$ is a clopen subset of the corresponding factor (see Proposition 2), hence $\mathcal{F}_\alpha$ is a centered system of clopen subsets in a cb-space X_α and so the intersection $\cap \{\overline{\pi_\alpha(U)} : U \in \mathcal{F}\}$ is non-void.Taking a point $x_\alpha \in \cap \{\overline{\pi_\alpha(U)} : U \in \mathcal{F}\}$ for every α , consider a point $z = (x_\alpha)$ in the product $\prod X_\alpha$; we shall show that $z \in \cap\{U : U \in \mathcal{F}\}$.

Really, if $z \notin \cap \{U : U \in \mathcal{F}\}$, then there exists a set $W \in \mathcal{F}$ that does not contain z . Using Proposition 3 we can find clopen sets $U_{\alpha_1} \subset X_{\alpha_1}, U_{\alpha_2} \subset X_{\alpha_2}, \ldots, U_{\alpha_n} \subset X_{\alpha_n}$ such that $z \in V$ and $V \cap W = \phi$ where $V = \prod U_\alpha$ and moreover $U_\alpha = X_\alpha$ for all α distinct from $\alpha_1, \ldots, \alpha_n$. Consider now the sets $V_{\alpha_i} = (X_{\alpha_i} \setminus U_{\alpha_i}) \times \prod_{\alpha \neq \alpha_i} X_\alpha$. It is easy to verify that $V_{\alpha_1} \cup \ldots \cup V_{\alpha_n} \cup V = X$ and as $\mathcal{F}$ is a clopen ultrafilter at least one of the sets $V_{\alpha_1}, \ldots, V_{\alpha_n}, V$ must belong to $\mathcal{F}$. On the other hand directly from the definition of the point z it is clear that no set V_{α_i} may belong to $\mathcal{F}$, besides V does not belong to $\mathcal{F}$ either because $W \in \mathcal{F}$ while $V \cap W = \phi$. This contradiction proves that z must belong to every $U \in \mathcal{F}$ and hence the clopen ultrafilter $\mathcal{F}$ converges. Now, using the corollary of Theorem 1 we conclude that the product $X = \prod X_\alpha$ is a cb-space.

In such a way Theorem 7 is proved.

§ 2. EXAMPLES

Example 1. All bicompact spaces are, obviously, clustered.

Example 2. All connected spaces are clustered.

Really, the only non void clopen subset of a connected space X is the whole X itself.

More generally, it is easy to notice that a space that can be represented as a finite union of its connected subspaces is also clustered.

Example 3. From Theorem 7 it follows that a product of a bicompact space and a connected space is clustered.

Example 4. Consider a subspace L of a unit interval $[0,1]$ defined by the equality $L = [0,1]\setminus\{\frac{1}{n}: n=1,2,\dots\}$. The space L is neither bicompact nor connected, but it is a cb-space.

Example 5. Modify the previous example taking for base on the set $[0,1]\setminus\{\frac{1}{n}: n = 1,2,\dots\}$ all sets open in L and also the sets of the form $V = \{0\}\cup(\bigcup(\frac{1}{n}, \frac{1}{n}+\varepsilon_n))$ where $0 < \varepsilon_n < \frac{1}{n^2}$ for every n . The space obtained in this way will be denoted by L'. It is easy to check that L' is a cb-space but it is neither bicompact nor connected. Moreover, the space L' is not first countable.

Example 6. According to Theorem 2 the property "to be clustered" is inherited by clopen sets. On the other hand, it is easy to notice, that closed sets do not inherit this property: the space N of natural numbers, being a closed subspace of the real line R, is not a cb-space.

It is natural, however, to ask a question about the heredity of the property "to be clustered" in the following refined form. Let X be a cb-space and Y its closed subspace. Can one affirm that every cover of Y by clopen in X sets has a finite subcover? The following example gives a negative answer to this question, too.

Consider a subspace X of the plane, defined by the equality $X = \bigcup X_n \cup \{b\}$ where every X_n is a set of points with the first coordinate equal to $\frac{1}{n}$ and the second belonging to the interval $[0,1]$ and b is the point $(0,1)$. It is not difficult to notice, that X is a cb-space -every clopen set, containing the point b must contain also almost all of X_n . Consider now the closed subspace $Y \subset X$ defined by the equality $Y = \{(1,0), (\frac{1}{2},0),\dots,(\frac{1}{n},0),\dots\}$.

It is clear that the family $\{X_n\}$ is a clopen cover of Y, but one cannot find a finite subcover in it.

One can also construct examples which show that the property to be clustered is not inherited by open subspaces, either.

Example 7. There exists a not clustered space X, which can be perfectly mapped onto a cb-space (even onto a connected space).

For such a space X one can take a subspace of the plane, defined by the equality $X = \bigcup X_n$ where $X_n = (n, [0,1])$ for all $n = 0, \pm 1, \pm 2, \ldots$. Obviously, the space X is not clustered. On the other hand, let a mapping f from X onto R be defined by the formula $f(n,x) = n + x$ for every point $(n,x) \in X$. It is clear that the mapping f is perfect (the preimage of every point $r \in R$ consists of either one or two points).

The same example shows us also that a space X which is not clustered may have for its quotient a connected space even in the case when all the equivalence classes are finite.

LITERATURE

[1] R.Engelking: Outline of General Topology.