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ON CONVERGENCE IN THE SPACE OF SUMMATIONS

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0. Introduction

In [17] and [8] the authors began to develop a general theory of summation on a state space X and a parameter space T. X and T are Hausdorff locally compact but non compact spaces. A summation $S = (\mu_t)_{t\in T}$ on X over T is a family of bounded Radon measures μ_t on X. The specific aspect of our theory is to describe the behavior of summations by weak convergence of the family $(\mu_t)_{t\in T}$ on suitable compactifications of the state space X. This theory gives a framework, which contains as a special case the classical summation by Toeplitz- matrices and has applications in stochastic processes [8]. Because of the last we name notions of summation in vocabulary close to that of stochastic processes. $\mathcal{M}(X)$ denotes the space of bounded Radon measures on X.

A summation $S = (\mu_t)_{t\in T}$ on X over T is called convergence preserving iff $S\text{-}\lim f := \lim_{t\to\infty} \mu_t(f)$ exists in R for every $f\in C_a(X)$, where $C_a(X)$ is the space of all continuous real functions on X having a limit at infinity. S is called permanent iff S is convergence preserving and $S\text{-}\lim f = \lim_{x\to\infty} f(x)$ for all $f\in C_a(X)$. S is called convergence generating iff $S\text{-}\lim f$ exists in R for all $f\in C_b(X)$. Here $C_b(X)$ is the space of all bounded real continuous functions on X. S is called core-contracting iff $\liminf_{x\to\infty} f(x) \leq \liminf_{t\to\infty} \mu_t(f) \leq$ $\leq \limsup_{t\to\infty} \mu_t(f) \leq \limsup_{x\to\infty} f(x)$ for all $f\in C_b(X)$.

The set of all summations on X over T we denote by $\mathcal{T}(X, T)$. $\mathcal{T}(X, T)$ becomes a vector space by $S_1 + S_2 := (\mu_t^1 + \mu_t^2)_{t\in T}$, where $S_i = (\mu_t^i)_{t\in T}$ and $\alpha \cdot S = (\alpha \cdot \mu_t)_{t\in T}$, where $S = (\mu_t)_{t\in T}$.

We like to introduce "natural" convergence structures in $\mathcal{T}(X, T)$ in such a way that the subsets of convergence preserving resp. permanent resp. convergence generating resp. core-contracting summations on X over T become closed. We shall see that convergence structures of the desired kind will not be topological. If we don't restrict ourselves to the linear subspace $\mathcal{T}_{ter}(X, T)$ of terminal uniformly bounded summations on X over T, the desired convergence structures are group convergence structures indeed and they are of $\mathcal{L}^*$- class but they fail to be compatible with scalar multiplication (Theorem 1.2.). A summation $S =$ $= (\mu_t)_{t\in T}$ on X over T is called terminal uniformly bounded iff there is a compact set $K \subseteq T$ such that $\{\|\mu_t\| ; t\in T\setminus K\}$ is bounded. It seems that in the classical case of matrix summation theory no analogous investigations were made.

In the last years in other mathematical topics suitable convergence structures, which are not topological, have been also introduced. E.g. Mikusinski [12] introduced a definition of convergence of sequences of Mikusinski operators. Urbanik [18] has shown that there is no topology satisfying the first axiom of countability such that topological convergence of usual sequences is equivalent with the Mikusinski convergence. E. F. Wagner [19] generalized the convergence of Mikusinski operators to nets. He shows that this convergence is not a topological one. In the paper [7] was treated the question when the locally uniform convergence of continuous functions is a topological one. Another mathematical topic of such kind is the modern theory of calculus in limitated vector spaces. See e.g. Binz [1] and Fröhlicher-Bucher [9].

1. The vector space of summations and closure properties of special summation types

1.1. Definition. Let $(S_\alpha)_{\alpha \in A} \subseteq \mathcal{T}(X,T)$ be a net and $S \in \mathcal{T}(X,T)$.

1. This net we will call weakly terminal convergent to S iff
   a) $(S_\alpha)_{\alpha \in A}$ simple converges in the weak topology $\sigma(\mathcal{M}(X), C_b(X))$ to S, i.e. $\mu_t^\alpha(f) \to \mu_t(f)$ for all $f \in C_b(X)$ and all $t \in T$.
   b) $(S_\alpha)_{\alpha \in A}$ converges uniformly at infinity in the weak topology $\sigma(\mathcal{M}(X), C_b(X))$; this means for every $f \in C_b(X)$ and $\varepsilon > 0$ there exists a compact set $K_{f,\varepsilon} \subseteq T$ such that $|\mu_t^\alpha(f) - \mu_t(f)| < \varepsilon$ for all $t \notin K_{f,\varepsilon}$ and all $\alpha \geqq \alpha_0 = \alpha_0(\varepsilon, f)$.
2. This net we will call weakly terminal uniformly convergent to S iff
   a) from above.
   b) $(S_\alpha)_{\alpha \in A}$ converges locally uniformly at infinity in the weak topology to S, i.e.
   there is a compact set $K \subseteq T$ such that for every $\varepsilon > 0$ and every $f \in C_b(X)$ $|\mu_t^\alpha(f) - \mu_t(f)| < \varepsilon$ holds for all $t \notin K$ and all $\alpha \geqq \alpha_0 = \alpha_0(\varepsilon, f)$.
3. This net we will call strongly terminal convergent to S iff
   a) from above.
   b) $(S_\alpha)_{\alpha \in A}$ converges uniformly at infinity in the strong topology (= norm topology) to S.
4. This net we will call strongly terminal uniformly convergent to S iff
   a) from above.
   b) $(S_\alpha)_{\alpha \in A}$ converges locally uniformly at infinity in the strong topology.

1.2. Theorem. In the space $\gamma(X,T)$ of all summations on X over T the following holds for the four types (1.1.) of convergence structures

(1) The convergence structures are of $\mathcal{L}^*$-class, i.e. they fullfill the following conditions:
   a) Uniqueness of limit.
   b) Subnets of convergent nets tend to the same limit.
   c) Urysohn axiom.

(2) The convergence structures (1.1.) are compatible with the additive structure of $\gamma(X,T)$ but not in general with scalar multiplication.

(3) The convergence structures (1.1.) are not compatible with the lattice structure.

(4) The convergence structures (1.1.) are in general not topological.

Proof.

Ad (1):

a) Uniqueness is obvious, because the simple convergence is unique.

b) Is clear.

c) Assume $(S_\alpha)_{\alpha\in A} \not\to S$.

We must show that there is a subnet of $(S_\alpha)_{\alpha\in A}$ which has no subnet converging to S .

We only consider the first type of convergence. Because the simple convergence is even topological, we can assume the following:
There is an $f \in C_b(X)$ and an $\varepsilon > 0$ such that for every compact set $K \subseteq T$ there exists a $t_K \notin K$ such that $(*)\ |\mu^\alpha_{t_K}(f) - \mu_{t_K}(f)| \geqq \varepsilon$ for all α of a confinal subset A_K of A .

Let $B := \{(\alpha, K)\ ;\ K$ compact subset of T and $\alpha \in A_K\}$.

B is directed as follows: $(\alpha, K) \prec (\tilde{\alpha}, \tilde{K})$ iff $\alpha < \tilde{\alpha}$ and $K \subseteq \tilde{K}$.

As the desired subnet we take $(S_{(\alpha,K)})_{(\alpha,K)\in B}$ with $S_{(\alpha,K)} := S_\alpha$.

Because of $(*)$ this subnet contains no subnet converging to S .

Ad (2):

This is straightforward with the help of triangle inequality for the additive structure.

From the example $S_n = S = (k \cdot \delta_k)_{k\in N}$, $X = T = N$, we can see that the scalar multiplication is not continuous with respect to the convergence, for $(1/n) \cdot S_n$ does not tend to zero.

Ad (3):

For every state space X the weak topology $\sigma(\mathcal{M}(X), C_b(X))$ and the norm topology never coincide. Namely, otherwise the unit ball in $\mathcal{M}(X)$ would be strongly compact, which is impossible for a state space. Therefore one has a net $(\mu^\alpha)_{\alpha\in A} \subseteq (X)$ which converges to zero weakly but not strongly, i.e. $\|\mu^{\alpha}\| \not\to 0$. Now we define on X over an arbitrary parameter space T the summations $S_{\alpha} := (\mu_t^{\alpha})_{t \in T}$, $S := (\mu_t)_{t \in T}$ with $\mu_t \equiv 0$ and $\mu_{t_0}^{\alpha} := \mu^{\alpha}$ for a fixed t_0 and $\mu_t^{\alpha} \equiv 0$ otherwise.

Ad (4):

Counterexample: $X := T := \{n - 1/2^m \; ; \; n \in N \text{ and } m \in N\}$. We take in $X = T$ the euclidean topology.

We will show that Kelley's theorem on iterated limits does not hold with respect to the convergence of type (4). For this we consider the following sequences of summations: $S_1, S_2, \ldots, S_n, \ldots \equiv 0$. For fixed $k \in N$ we take $S_{n(k)} = (\mu_t^{n(k)})_{t \in T}$, where $\mu_t^{n(k)} \equiv 0$ for all $t \notin \{k - 1/2^m \, , \, m \geq 1\}$.

And for $t \in \{k - 1/2^m \; ; \; m \geq 1\}$ the measure is distributed in the following manner:

For $n = 1$, $\mu_t^{1(k)}$ has mass $1/2^m$ in the point $x = k - 1/2^m$ for $x \geq t$.

For $n = 2$, the point masses are moved by one step to the right, and so on.

Then $S_{n(k)} \to S_k \equiv 0$ for $n \to \infty$.

But the iterated net $S_{[k,j]} := S_{n(k)}$ with $j \in N^{\aleph_0}$ and $n(k) =$ k-coordinate of j does not converge to zero.

1.3. <u>Theorem.</u> The vector space $\mathcal{T}_{ter}(X,T)$ of all terminal uniformly bounded summations on X over T for all convergence structures always becomes a convergence vector space.

The set $Pr(X,T)$ of all terminal uniformly bounded convergence preserving summations on X over T forms in all four convergence structures a closed linear subspace of $\mathcal{T}_{ter}(X,T)$.

The set $Per(X,T)$ of all terminal uniformly bounded permanent summations on X over T forms in all four convergence structures a closed convex subset of $\mathcal{T}_{ter}(X,T)$.

The set $Per(X,T)$ of all terminal uniformly bounded permanent summations on X over T forms in all four convergence structures a closed convex subset of $\mathcal{T}_{ter}(X,T)$.

The set $Gen(X,T)$ of all terminal uniformly bounded convergence generating summations on X over T forms for all four convergence structures a closed linear subspace of $\mathcal{T}_{ter}(X,T)$.

Proof.

1) Because of Theorem 1.2. $\mathcal{T}(X,T)$ is a convergence group for the four convergence structures. $\mathcal{T}_{ter}(X,T)$ is even a convergence vector space. For this one has to consider the following estimation:

$$|\lambda_{\alpha} \cdot \mu_t^{\alpha}(f) - \lambda \cdot \mu_t(f)| \leq |\lambda_{\alpha}| \cdot |\mu_t^{\alpha}(f) - \mu_t(f)| + |\mu_t(f)| \cdot |\lambda_{\alpha} - \lambda|$$

for the case $\lambda_\alpha \to \lambda$ and $S_\alpha = (\mu_t^\alpha)$, $S = (\mu_t)$.

2) It is straightforward to check the linear structure of Pr(X,T) and Gen(X,T) and the convexity of Per(X,T) and Cor(X,T) .

3) For the closedness we may restrict ourselves to the weakest convergence structure.

Let us check for example the cases Pr(X,T) and Cor(X,T) . Let $S_\alpha = (\mu_t^\alpha) \in Pr(X,T)$ and $S \in \mathcal{T}_{ter}(X,T)$ and $S_\alpha \to S$ weakly terminal. We have to show: $\lim_{t\to\infty} \mu_t(f)$ exists in R for each $f \in C_a(X)$. Let $f \in C_a(X)$.
We get $|\mu_t(f) - \mu_{t'}(f)| \leqq |\mu_t(f) - \mu_t^\alpha(f)| + |\mu_t^\alpha(f) - \mu_{t'}^\alpha(f)| + |\mu_{t'}^\alpha(f) - \mu_{t'}(f)|$. By b) there exists for every $\varepsilon > 0$ a compact set ${}_\alpha K_{f,\varepsilon} \subseteq T$ such that $|\mu_t(f) - \mu_t^\alpha(f)| < \varepsilon/_3$ and $|\mu_{t'}(f) - \mu_{t'}^\alpha(f)| < \varepsilon/_3$ for $t,t' \in T \setminus K_{f,\varepsilon}$ and $\alpha \geqq \alpha_0$. We take such an α . S is convergence preserving. Therefore we get a compact set $\tilde{K}_{\varepsilon,f} \subseteq T$ such that $|\mu_{t'}^\alpha(f) - \mu_t^\alpha(f)| < \varepsilon/_3$ for $t,t' \in T \setminus \tilde{K}_{\varepsilon,f}$.
Thus $|\mu_t(f) - \mu_{t'}(f)| < \varepsilon/_3 + \varepsilon/_3 + \varepsilon/_3 = \varepsilon$ for all $t, t' \in T \setminus (K_{f,\varepsilon} \cup \tilde{K}_{\varepsilon,f})$. Therefore $S \in Pr(X,T)$.
Now let $S_\alpha = (\mu_t^\alpha) \in Cor(X,T)$, $S \in \mathcal{T}_{ter}(X,T)$ and $S_\alpha \to S$ weakly terminal. For $f \in C_b(X)$ we put $b := \liminf_{x\to\infty} f(x)$ and $a := \liminf_{t\to\infty} \mu_t(f) := \sup_{K\subseteq T} (\inf_{t\notin K} \mu_t(f))$, K compact .
Let $\varepsilon > 0$. Then there is a compact set $K_\varepsilon \subseteq T$ such that $\inf_{t\notin K} \mu_t(f) > a - \varepsilon/_2$. Because of condition b) we can assume without loss of generality $|\mu_t(f) - \mu_t^\alpha(f)| < \varepsilon/_2$ for all $t \notin K_\varepsilon$ and almost all α . We easily conclude $|\inf_{t\notin K} \mu_t(f) - \inf_{t\notin K_\varepsilon} \mu_t^\alpha(f)| \leqq \varepsilon/_2$ for almost all α .
Now we have $\inf_{t\notin K_\varepsilon} \mu_t^\alpha(f) \leqq \inf_{t\notin K_\varepsilon} \mu_t(f) + \varepsilon/_2$. From $S_\alpha \in Cor(X,T)$ we can assume $b - \varepsilon/_2 < \inf_{t\notin K} \mu_t^\alpha(f)$. Therefore we get

$$b - \varepsilon/_2 \leqq \inf_{t\notin K_\varepsilon} \mu_t^\alpha(f) \leqq \inf_{t\notin K_\varepsilon} \mu_t(f) + \varepsilon/_2 \leqq a + \varepsilon/_2 . \; b \leqq a + \varepsilon .$$

In a dual manner we get the rest of the proof.

1.4. <u>Theorem</u>. The vector space of all uniformly bounded summations $\mathcal{T}_b(X,T)$ on X over T equipped with the norm $\|S\| := \sup_{t\in T} \|\mu_t\|$, $S = (\mu_t)_{t\in T} \in \mathcal{T}_b(X,T)$, is a Banach space.
The sets $Pr(X,T) \cap \mathcal{T}_b(X,T)$, $Per(X,T) \cap \mathcal{T}_b(X,T)$, $Cor(X,T) \cap \mathcal{T}_b(X,T)$ and $Gen(X,T) \cap \mathcal{T}_b(X,T)$ are closed.

Proof. $\mathcal{T}_b(X,T)$ with the norm $\|S\| := \sup_{t\in T} \|\mu_t\|$ equals the Banach space $l^\infty(T, \mathcal{M}(X))$. The rest follows from 1.3.

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