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SUBBASE STRUCTURES IN NEARNESS SPACES

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The aim of this note is to introduce the notion of a subbase for a Nearness space.

NEARNESS SPACES were introduced by HERRLICH in [9,10] for the following reasons.

a) Unification of the theories of proximity, uniformity, contiguity, merotopic spaces; cf. e.g. [4,7,8,11,12,14].
b) To give a richer structure than in topology in which uniform continuity, Cauchy filters, even covers etc. can be expressed without loosing essential parts of general topology.
c) The category of Nearness spaces and N-morphisms is a little smoother than the category *Top*. Especially product constructions are nicer.

For a more extended motivation and a bibliography we refer to [10].

SUBBASES are important in general topology, because several notions, characterizations and constructions are given in terms of subbases. We mention for instance:

1) Construction of product spaces. (The collection of inverse images, with respect to projections, of open subsets in the coördinate spaces is a subbase for the product topology.)
2) URYSOHN's metrization theorem [15]: A regular separable T_1 space is metrizable iff it has a countable (sub)base.
3) A space is generalized orderable iff it has a T_1 subbase consisting of two nests [3].
4) Alexander's theorem. A space is compact iff it is compact relative to its subbases [1].
5) The DE GROOT theory on superextensions and supercompactness and the DE GROOT & AARTS compactification method by means of linked systems chosen from (weakly) normal subbases [5,6,16].

With the definition of an N-subbase which is exposed here we will adapt those subjects for N-spaces.

DEFINITION OF THE NEARNESS SPACE (X,μ). [HERRLICH] [10]. Let X be set and let $\mu \subset P(P(X))$, then μ is a *collection of uniform covers* in an N-space iff μ satisfies the axioms:

(i) If A is refined by some $B \in \mu$ then $A \in \mu$.

(ii) Members of μ are covers of X.

(iii) $\{X\} \in \mu$.

(iv) $A, B \in \mu$ then $A \wedge B = \{A \cap B \mid A \in A;\ B \in B\} \in \mu$.

(v) $A \in \mu$ then $\{\mathrm{Int}(A) \mid A \in A\} \in \mu$

in which $\mathrm{Int}(A) = \{x \mid \{A, X\setminus\{x\}\} \in \mu\}$.

The interior operator claimed in (v) defines a topology on the set X, compatible with the nearness structure μ. This topology satisfies the following axiom:

$$(R_0) \qquad \forall x,y \in X: \qquad x \in Cl_X(y) \Longleftrightarrow y \in Cl_X(x).$$

An N-space is *topological* iff all open covers of this topology are in μ.
An N-space is *contiqual* iff every cover in μ has a finite refinement which is in μ.
An N-space is *compact* iff it is topological and contiqual.
An N-space is *uniform* iff every cover in μ has a star refinement in μ.
An N-*morphism* between (X,μ_X) and (Y,μ_Y) is a set function $f: X \to Y$ such that:

$$\forall A \in \mu_Y: \qquad \{f^{\leftarrow}[A] \mid A \in A\} \in \mu_X.$$

N-spaces are completely determined by the set of all *open* covers in μ, and for the sequel we restrict ourselves to open covers of X.

DEFINITION OF AN N-SUBBASE. An *N-subbase* σ for a nearness structure on X is a collection of covers of X which satisfies:

$\forall\, x \in S \in \mathcal{S} \in \sigma$:

$\exists\, \mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ in σ such that $\{X\setminus\{x\}, S\}$ is

refined by $\mathcal{S}_1 \wedge \mathcal{S}_2 \wedge \ldots \wedge \mathcal{S}_n$.

The underlying topological space is constructed by taking

$$\mathcal{S}_\sigma = \{S \mid S \in \mathcal{S} \in \sigma\}$$

as an open subbase.
We obtain the N-space defined by the N-subbase by taking all covers of X which are refined by finite $\wedge$-intersections of covers in σ.

Every collection of covers γ of X can be extended to an N-subbase σ. We put:

$$\sigma = \gamma \cup \{\{X\setminus\{x\}, C\} \mid x \in C \in \mathcal{C} \in \gamma\}.$$

A *cluster* in an N-subbase or in an N-space is a maximal collection of open sets which does not contain an admissible cover. Extensions of N-spaces are constructed on the collection of all clusters.

If U is an open set in X then U^+ is the collection of all clusters which do not contain U. We obtain a new N-space by taking extensions of admissible covers:

$$\sigma^+ = \{\mathcal{S}^+ \mid \mathcal{S} \in \sigma\}$$

in which

$$\mathcal{S}^+ = \{U^+ \mid U \in \mathcal{S}\}.$$

For instance, in contigual spaces there are sufficiently many clusters to obtain well defined extensions.

APPLICATIONS

1) We obtain a subbase for the product N-structure of a collection of N-spaces if we take all inverse images under projections of the open covers in the coordinate N-spaces.
2) An N-space is metrizable iff it is uniform and it has a countable N-subbase. (ALEXANDROFF-URYSOHN [2] adapted in [10]).
3) An N-space is generalized orderable iff it has an N-subbase σ separating points, such that every cover in σ consists of two elements and S_σ consists of two nests. ([3], adapted).

   Moreover, the cluster-extension of such an N-space is compact and ordered, and all the order-preserving compactifications of the underlying topological space can be obtained in this way.
4) An N-space is contigual iff it has an N-subbase consisting of finite covers. (ALEXANDER [1] adapted).
5) An N-space is *supercontigual* iff it has an N-subbase consisting of two-element covers. N-spaces which are both topological and supercontigual are *supercompact*. The cluster extension of a supercontigual N-space is supercompact. The closure of the underlying space in the extension is a compactification. If the underlying N-subbase separates points and subbase emembers and satisfies some condition of weak normality (in [7] screening) then this compactification is a Hausdorff compactification. This is an adapation of the DE GROOT theory on superextensions [5,6,16].

HAMBURGER [7] showed that all T_2-compactifications can be obtained by means of strong preproximities. There is a canonical way to define an N-subbase for a supercontigual space from a preproximity. A pair {A,B} is in σ iff $\{X\backslash A, X\backslash B\}$ is not "near" in the preproximity. Following this modification HAMBURGER's paper shows that all T_2-compactifications can be derived from cluster extensions.

However, the question whether all T_2-compactifications can be derived directly by DE GROOT's method is still open.

Recent results of VAN MILL suggest a positive answer [13].

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