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IDEALS OF OPERATORS ON BANACH SPACES AND NUCLEAR LOCALLY CONVEX SPACES

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Many theorems about nuclear locally convex spaces can be proved without using special properties of nuclearity. We only need the fact that nuclear operators form an ideal. Since the same is true for Schwartz spaces in what follows we present a general theory of $\mathfrak{S}$-spaces which are related to an arbitrary ideal $\mathfrak{S}$ of operators.

1. Ideals of operators on Banach spaces

Let $\mathfrak{L}$ be the class of all bounded linear operators between arbitrary Banach spaces. The set of operators $S \in \mathfrak{L}$ which map from the Banach space E into the Banach space F is denoted by $\mathfrak{L}(E, F)$.

A subclass $\mathfrak{S}$ of $\mathfrak{L}$ is called an *ideal* if for the sets

$$\mathfrak{S}(E, F) := \mathfrak{S} \cap \mathfrak{L}(E, F)$$

the following axioms are satisfied:

(I_1) If $S \in \mathfrak{L}(E, F)$ and $\dim S(E) < \infty$ then $S \in \mathfrak{S}(E, F)$.
(I_2) If $S_1, S_2 \in \mathfrak{S}(E, F)$ then $S_1 + S_2 \in \mathfrak{S}(E, F)$.
(I_3) If $S \in \mathfrak{S}(E, F)$ and $R \in \mathfrak{L}(F, G)$ then $RS \in \mathfrak{S}(E, G)$.
(I_4) If $T \in \mathfrak{L}(E, F)$ and $S \in \mathfrak{S}(F, G)$ then $ST \in \mathfrak{S}(E, G)$.

The class $\mathfrak{F}$ of all bounded linear operators with finite dimensional range is the smallest ideal.

2. Locally convex spaces of type $\mathfrak{S}$

Let p be a seminorm on the linear space E. We denote by $E(p)$ the quotient space $E/N(p)$, $N(p) := \{x \in E : p(x) = 0\}$, with the elements $x(p) := x + N(p)$ and the norm $\|x(p)\| := p(x)$. The Banach space $\tilde{E}(p)$ will be the complete hull of $E(p)$.

If p and q are seminorms such that $q(x) \leqq c\, p(x)$ for all $x \in E$, where c is a constant, we write $q \prec p$. Then $N(p) \subset N(q)$ and a bounded linear operator

$E(p, q)$ from $E(p)$ onto $E(q)$ is defined by

$$E(p, q)\, x(p) := x(q)\,.$$

$\tilde{E}(p, q)$ will be the unique extension of $E(p, q)$ to a bounded linear operator from $\tilde{E}(p)$ into $\tilde{E}(q)$.

A system P of seminorms on a linear space E is called *saturated* if the following axioms are satisfied:

(P_1) If $p \in P$ and $q \prec p$ then $q \in P$.
(P_2) If $p_1, p_2 \in P$ then there exists $p \in P$ such that $p_1 \prec p$, $p_2 \prec p$.
(P_3) If $x \in E$ such that $p(x) = 0$ for all $p \in P$ then $x = o$.

A subsystem P_0 of a saturated system P of seminorms is a *basis* if for each $p \in P$ there is $p_0 \in P_0$ such that $p \prec p_0$.

A *locally convex space* $[E, P]$ is a linear space E with a saturated system P of seminorms. Let $\mathfrak{S}$ be an ideal of operators, then the locally convex space $[E, P]$ is called of *type* $\mathfrak{S}$, or $\mathfrak{S}$-*space*, if for some basis system P_0 of seminorms the following property holds:

(S) For every $q \in P_0$ there exists $p \in P_0$ such that $q \prec p$ and $\tilde{E}(p, q) \in \mathfrak{S}$.

It is easy to see that for a locally convex space of type $\mathfrak{S}$ every basis system of seminorms has property (S).

The class of locally convex $\mathfrak{S}$-spaces is denoted by $L_{\mathfrak{S}}$.

3. Examples

3.1. The ideal of operators with finite dimensional range. A locally convex space $[E, P]$ is of type $\mathfrak{F}$ if and only if every Banach space $\tilde{E}(p)$, $p \in P$, has finite dimension, i.e., $[E, P]$ is a locally convex space with the weak topology.

3.2. The ideal of compact operators. An operator $S \in \mathfrak{L}(E, F)$ is called compact if $S(U_E)$, $U_E := \{x \in E : \|x\| \leqq 1\}$, is a precompact subset of F. The class $\mathfrak{K}$ of compact operators is the oldest known ideal. The Schwartz spaces are the locally convex spaces of type $\mathfrak{K}$ (cf. [4]).

3.3. The ideal of L_p-factorable operators. An operator $S \in \mathfrak{L}(E, F)$ is called L_p-factorable, $1 \leqq p \leqq \infty$, if there exists a measure space $[\Omega, B, \mu]$ such that

$$S : E \xrightarrow{A} L_p[\Omega, B, \mu] \xrightarrow{Y} F\,,$$

where $A \in \mathfrak{L}(E, L_p)$ and $Y \in \mathfrak{L}(L_p, F)$ (cf. [7], [13]). The class $\mathfrak{L}_p$ of L_p-factorable operators is an ideal. Banach spaces of type $\mathfrak{L}_p$ were considered by J. Lindenstrauss,

A. Pełczyński and H. Rosenthal (cf. [8], [9]). The case $p = 2$ is of special interest since a locally convex space $[E, P]$ is of type $\mathfrak{L}_2$ if and only if there exists a basis system P_0 of seminorms such that every $p \in P_0$ can be obtained from a semi-scalarproduct $(.,.)_p$ by $p(x) = (x, x)_p^{1/2}$.

3.4. The ideal of nuclear operators. An operator $S \in \mathfrak{L}(E, F)$ is called nuclear if there exist functionals $a_1, a_2, \ldots \in E'$ and elements $y_1, y_2, \ldots \in F$ such that

$$Sx = \sum_k \langle x, a_k \rangle\, y_k \quad \text{for all} \quad x \in E$$

and

$$\sum_k \|a_k\|\, \|y_k\| < \infty .$$

The class $\mathfrak{N}$ of nuclear operators is an ideal. The nuclear locally convex spaces are the locally convex spaces of type $\mathfrak{N}$ (cf. [6], [11]).

3.5. The ideal of absolutely p-summing operators. An operator $S \in \mathfrak{L}(E, F)$ is called absolutely p-summing, $0 < p < \infty$, if there exists a constant $c \geqq 0$ such that for every finite system of elements $x_1, \ldots, x_m \in E$ the inequality

$$\{\sum_k \|Sx_k\|^p\}^{1/p} \leqq c \sup_{\|a\| \leqq 1} \{\sum_k |\langle x_k, a \rangle|^p\}^{1/p}$$

holds. The class $\mathfrak{P}_p$ of absolutely p-summing operators is an ideal, and we obtain the nuclear locally convex spaces as the locally convex spaces of type $\mathfrak{P}_p$ (cf. [12]).

3.6. The ideal of $\mathfrak{S}_p^{\mathrm{app}}$-operators. The approximation numbers of an operator $S \in \mathfrak{L}(E, F)$ are defined by

$$s_k(S) := \inf \{\|S - A\| : A \in \mathfrak{F}(E, F), \dim A(E) < k\}$$

for $k = 1, 2, \ldots$. The operators $S \in \mathfrak{L}$ with

$$\sum_k s_k(S)^p < \infty$$

form the ideal $\mathfrak{S}_p^{\mathrm{app}}$, $0 < p < \infty$. The nuclear locally convex spaces are the locally convex spaces of type $\mathfrak{S}_p^{\mathrm{app}}$ (cf. [11], [14]).

3.7. The ideal of $\mathfrak{S}_0^{\mathrm{app}}$-operators. Let

$$\mathfrak{S}_0^{\mathrm{app}} := \bigcap_{p>0} \mathfrak{S}_p^{\mathrm{app}} ,$$

then the locally convex spaces of type $\mathfrak{S}_0^{\mathrm{app}}$ are the so-called strictly nuclear locally convex spaces (cf. [1], [2], [10]).

4. Ideals of sequences

Let $\mathfrak{l}_\infty$ be the ring of all bounded sequences (σ_k). A subset $\mathfrak{s}$ of $\mathfrak{l}_\infty$ is called an *ideal* if the following axioms are satisfied:

(E_1) If $\{k : \sigma_k \neq 0\}$ is finite then $(\sigma_k) \in \mathfrak{s}$.
(E_2) If $(\sigma_k^{[1]}), (\sigma_k^{[2]}) \in \mathfrak{s}$ then $(\sigma_k^{[1]} + \sigma_k^{[2]}) \in \mathfrak{s}$.
(E_3) If $(\sigma_k) \in \mathfrak{s}$ and $(\varrho_k) \in \mathfrak{l}_\infty$ then $(\varrho_k \sigma_k) \in \mathfrak{s}$.
(E_4) If $(\sigma_k) \in \mathfrak{s}$ and π is a permutation of the natural numbers then $(\sigma_{\pi(k)}) \in \mathfrak{s}$.

The connection between ideals of operators and ideals of sequences is described in the following (cf. [3], [5])

Theorem. *Let $\mathfrak{S}$ be an ideal of operators. Then the set $\mathfrak{s}$ of sequences $(\sigma_k) \in \mathfrak{l}_\infty$ such that $D \in \mathfrak{S}(l_2, l_2)$, where $D(\xi_k) := (\sigma_k \xi_k)$, is an ideal. Moreover, $S \in \mathfrak{S}(l_2, l_2)$ if and only if $(s_k(S)) \in \mathfrak{s}$.*

An operator $S \in \mathfrak{L}(E, F)$ is called *l_2-factorable* if there are operators $A \in \mathfrak{L}(E, l_2)$ and $Y \in \mathfrak{L}(l_2, F)$ such that $S = YA$. The class $\mathfrak{H}$ of l_2-factorable operators is an ideal.

Theorem. *Let $\mathfrak{S}$ be an ideal of operators such that $\mathfrak{S} \subset \mathfrak{H}$. Then the class $L_{\mathfrak{S}}$ is uniquely determined by the corresponding ideal $\mathfrak{s}$ of sequences.*

Now we state some lemmas.

Lemma 1. *Let $\mathfrak{s}$ be an ideal of sequences and let $m = 1, 2, \ldots$. If*

$$\sigma_k^{(m)} := \sigma_i \quad \text{for} \quad k = (i-1)\,m + j\,, \quad i = 1, 2, \ldots, j = 1, \ldots, m\,,$$

then $(\sigma_k) \in \mathfrak{s}$ implies $(\sigma_k^{(m)}) \in \mathfrak{s}$.

Proof. We obtain the result as follows:

$$(\sigma_1, \sigma_2, \sigma_3, \ldots) \in \mathfrak{s}\,,$$
$$(\sigma_1, 0, \sigma_3, \ldots) \in \mathfrak{s}\,,$$
$$(\underbrace{\sigma_1, \ldots, 0, 0}_{m}; \underbrace{0, 0, \ldots, 0, 0}_{m}; \underbrace{\sigma_3, \ldots, 0, 0}_{m}; \ldots) \in \mathfrak{s}\,,$$
$$(0, 0, \ldots, \sigma_1\,; 0, 0, \ldots, 0, 0; 0, 0, \ldots, \sigma_3\,; \ldots) \in \mathfrak{s}\,,$$
$$(\sigma_1, \ldots, \sigma_1\,; 0, 0, \ldots, 0, 0; \sigma_3, \ldots, \sigma_3\,; \ldots) \in \mathfrak{s}\,,$$
$$(0, 0, \ldots, 0, 0; \sigma_2, \ldots, \sigma_2\,; 0, 0, \ldots, 0, 0; \ldots) \in \mathfrak{s}\,,$$
$$(\sigma_1, \ldots, \sigma_1\,; \sigma_2, \ldots, \sigma_2\,; \sigma_3, \ldots, \sigma_3\,; \ldots) \in \mathfrak{s}\,.$$

Lemma 2. *Let $\mathfrak{S}$ and $\mathfrak{s}$, resp. $\mathfrak{T}$ and $\mathfrak{t}$, be corresponding ideals of operators or sequences. Then the following conditions are equivalent:*

(1) *If* $(\sigma_k^{[1]}), \ldots, (\sigma_k^{[m]}) \in \mathfrak{s}$ *then* $(\sigma_k^{[1]} \ldots \sigma_k^{[m]}) \in \mathfrak{t}$.
(2) *If* $S_1, \ldots, S_m \in \mathfrak{S}(l_2, l_2)$ *then* $S_1 \ldots S_m \in \mathfrak{T}(l_2, l_2)$.

Proof. (1) → (2): If $k = (i-1)m + j$, $i = 1, 2, \ldots, j = 1, \ldots, m$, then

$$s_k(S_1 \ldots S_m) \leqq s_{(i-1)m+1}(S_1 \ldots S_m) \leqq s_i(S_1) \ldots s_i(S_m) = s_k^{(m)}(S_1) \ldots s_k^{(m)}(S_m).$$

Since Lemma 1 implies $(s_k^{(m)}(S_j)) \in \mathfrak{s}$ for $j = 1, \ldots, m$ we obtain $(s_k(S_1 \ldots S_m)) \in \mathfrak{t}$. Consequently, $S_1 \ldots S_m \in \mathfrak{T}(l_2, l_2)$.

(2) → (1): The proof is left to the reader.

Lemma 3. *Let* $\mathfrak{S}$ *and* $\mathfrak{s}$ *be corresponding ideals of operators or sequences. Then the following conditions are equivalent:*

(M) *If* $S_1, S_2, \ldots \in \mathfrak{S}(l_2, l_2)$ *then there exist operators* $X_1, X_2, \ldots \in \mathfrak{L}(l_2, l_2)$, $B_1, B_2, \ldots \in \mathfrak{L}(l_2, l_2)$, *and* $S \in \mathfrak{S}(l_2, l_2)$ *such that*

$$S_h = B_h S X_h \quad for \quad h = 1, 2, \ldots.$$

(m) *If* $(\sigma_k^{[1]}), (\sigma_k^{[2]}), \ldots \in \mathfrak{s}$ *then there exist positive numbers* $\varrho_1, \varrho_2, \ldots$ *and* $(\sigma_k) \in \mathfrak{s}$ *such*

$$|\sigma_k^{[h]}| \leqq \varrho_h |\sigma_k| \quad for \quad k = 1, 2, \ldots.$$

Remark. Condition (M) is satisfied for every ideal $\mathfrak{S}$ of operators which is complete with respect to a quasinorm.

5. Equivalent ideals of operators

We have seen that the class of nuclear locally convex spaces can be obtained from different ideals, e.g. $\mathfrak{N}$, $\mathfrak{P}_p$, and $\mathfrak{S}_p^{app}$, $0 < p < \infty$. Consequently, if $\mathfrak{S}$ and $\mathfrak{T}$ are ideals of operators, it is useful to know necessary and sufficient conditions for the coincidence of locally convex spaces of type $\mathfrak{S}$ and $\mathfrak{T}$.

Theorem. *Let* $\mathfrak{S}$ *and* $\mathfrak{T}$ *be ideals of operators. If there exists a natural number n such that*

$$S_1 \in \mathfrak{S}(E_1, E_0), \ldots, S_n \in \mathfrak{S}(E_n, E_{n-1}) \quad implies \quad S_1 \ldots S_n \in \mathfrak{T}(E_n, E_0)$$

then $L_{\mathfrak{S}} \subset L_{\mathfrak{T}}$.

Now we prove a partial converse.

Theorem. *Let* $\mathfrak{S}$ *and* $\mathfrak{T}$ *be ideals of operators such that* $L_{\mathfrak{S}} \subset L_{\mathfrak{T}}$. *If* $\mathfrak{S}$ *satisfies condition* (M) *and* $\mathfrak{S} \subset \mathfrak{H}$ *then there exists a natural number n such that*

$$S_1 \in \mathfrak{S}(E_1, E_0), \ldots, S_n \in \mathfrak{S}(E_n, E_{n-1}) \quad implies \quad S_1 \ldots S_n \in \mathfrak{T}(E_n, E_0).$$

Proof. (1) In the first step we consider $(\sigma_k) \in \mathfrak{s}$ such that

$$\sigma_1 \geqq \sigma_2 \geqq \ldots > 0 .$$

Let

$$E := \{x = (\xi_k) : \sum_k \sigma_k^{-2l} |\xi_k|^2 < \infty,\ l = 1, 2, \ldots\}$$

and

$$p_l(x) := \{\sum_k \sigma_k^{-2l} |\xi_k|^2\}^{1/2}, \quad l = 1, 2, \ldots .$$

Moreover, we define by

$$D(\xi_k) := (\sigma_k \xi_k)$$

and

$$I_l(\xi_k) := (\sigma_k^{-l} \xi_k), \quad l = 1, 2, \ldots$$

an operator $D \in \mathfrak{S}(l_2, l_2)$ and isomorphisms $I_l \in \mathfrak{L}(\tilde{E}(p_l), l_2)$. Consequently, it follows from the commutative diagram

$$\begin{array}{ccc} \tilde{E}(p_{l+1}) & \xrightarrow{\tilde{E}(p_{l+1}, p_l)} & \tilde{E}(p_l) \\ {\scriptstyle I_{l+1}} \downarrow & & \uparrow {\scriptstyle I_l^{-1}} \\ l_2 & \xrightarrow[D]{} & l_2 \end{array}$$

that the locally convex space $[E, (p_l)]$ is of type $\mathfrak{S}$. Since $[E, (p_l)]$ is also of type $\mathfrak{T}$ there exists a natural number m such that $\tilde{E}(p_m, p_0) \in \mathfrak{T}$. Therefore, the commutative diagram

$$\begin{array}{ccc} \tilde{E}(p_m) & \xrightarrow{\tilde{E}(p_m, p_0)} & \tilde{E}(p_0) \\ {\scriptstyle I_m^{-1}} \uparrow & & \downarrow {\scriptstyle I_0} \\ l_2 & \xrightarrow[D^m]{} & l_2 \end{array}$$

implies that $D^m \in \mathfrak{T}(l_2, l_2)$. Hence $(\sigma_k^m) \in \mathfrak{t}$.

(2) Let us suppose that for every natural number $h = 1, 2, \ldots$ there exist $(\sigma_k^{[h,1]}), \ldots, (\sigma_k^{[h,h]}) \in \mathfrak{s}$ such that $(\sigma_k^{[h,1]} \ldots \sigma_k^{[h,h]}) \notin \mathfrak{t}$. Since $\mathfrak{s}$ satisfies condition (m) we find positive numbers $\varrho_{h,i}$, $h = 1, 2, \ldots, i = 1, \ldots, h$, and $(\sigma_k) \in \mathfrak{s}$ such that

$$|\sigma_k^{[h,i]}| \leqq \varrho_{h,i} |\sigma_k| .$$

Without loss of generality we may assume that

$$|\sigma_1| \geqq |\sigma_2| \geqq \ldots > 0 .$$

Consequently, $(\sigma_k^m) \in \mathfrak{t}$, where m is a natural number. Finally, we obtain $(\sigma_k^{[m,1]} \dots \sigma_k^{[m,m]}) \in \mathfrak{t}$. Contradiction.

(3) Since there is a natural number m such that

$$(\sigma_k^{[1]}), \dots, (\sigma_k^{[m]}) \in \mathfrak{s} \quad \text{implies} \quad (\sigma_k^{[1]} \dots \sigma_k^{[m]}) \in \mathfrak{t}$$

it follows from Lemma 2 that

$$S_1, \dots, S_m \in \mathfrak{S}(l_2, l_2) \quad \text{implies} \quad S_1 \dots S_m \in \mathfrak{T}(l_2, l_2)\,.$$

We put $n = 2m + 1$. If $S_i \in \mathfrak{S}(E_i, E_{i-1})$, $i = 1, \dots, n$, then we find factorizations $S_i = X_i A_i$, $A_i \in \mathfrak{L}(E_i, l_2)$ and $X_i \in \mathfrak{L}(l_2, E_{i-1})$. Consequently,

$$S_1 \dots S_n = X_1 \dots (A_{2j-1} S_{2j} X_{2j+1}) \dots A_{2m+1} \in \mathfrak{T}(E_n, E_0)\,.$$

6. Permanence properties

Without proofs we state some permanence properties.

Proposition. *The complete hull of a locally convex $\mathfrak{S}$-space is of type $\mathfrak{S}$.*

Proposition. *The product of an arbitrary set of locally convex $\mathfrak{S}$-spaces is of type $\mathfrak{S}$.*

An ideal $\mathfrak{S}$ of operators is called *injective* if the following axiom is satisfied (cf. [19]):

(J) Let $J \in \mathfrak{L}(F, F_0)$ be an injection (one-to-one operator with closed range) then $S \in \mathfrak{L}(E, F)$ and $JS \in \mathfrak{S}(E, F_0)$ imply $S \in \mathfrak{S}(E, F)$.

The ideals $\mathfrak{K}$, $\mathfrak{L}_2$, $\mathfrak{H}$, $\mathfrak{S}_0^{\mathrm{app}}$, and $\mathfrak{P}_p$, $0 < p < \infty$, are injective.

Proposition. *Let $\mathfrak{S}$ be an injective ideal of operators or let $\mathfrak{S} \subset \mathfrak{L}_2$. Then every subspace of a locally convex $\mathfrak{S}$-space is of type $\mathfrak{S}$.*

An ideal $\mathfrak{S}$ of operators is called *surjective* if the following axiom is satisfied:

(Q) Let $Q \in \mathfrak{L}(E_0, E)$ be a surjection (operator onto E) then $S \in \mathfrak{L}(E, F)$ and $SQ \in \mathfrak{S}(E_0, F)$ imply $S \in \mathfrak{S}(E, F)$.

The ideals $\mathfrak{K}$, $\mathfrak{L}_2$, $\mathfrak{H}$, and $\mathfrak{S}_0^{\mathrm{app}}$ are surjective.

Proposition. *Let $\mathfrak{S}$ be an surjective ideal of operators or let $\mathfrak{S} \subset \mathfrak{L}_2$. Then every quotient space of a locally convex $\mathfrak{S}$-space is of type $\mathfrak{S}$.*

7. Locally convex spaces of type $\mathfrak{L}_p$

It is easy to see that

$$L_{\mathfrak{N}} \subset L_{\mathfrak{L}_p} \quad \text{for all} \quad p \in [1, \infty] .$$

On the other side, from the results of [15] follows that

$$L_{\mathfrak{N}} = L_{\mathfrak{L}_1} \cap L_{\mathfrak{L}_p} \quad \text{for all} \quad p \in (1, \infty]$$

and

$$L_{\mathfrak{N}} = L_{\mathfrak{L}_\infty} \cap L_{\mathfrak{L}_p} \quad \text{for all} \quad p \in [1, \infty) .$$

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