ON DESCRIPTIVE CLASSIFICATION OF FUNCTIONS

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In this note, an extension of the classic descriptive theory of sets and functions is developed which makes possible, in principle, a classification of all discontinuous functions on any topological space admitting of a one-to-one continuous mapping onto a separable metrizable space. The basic device are filters on countably infinite sets and their types. It turns out e.g. that every Baire class is generated by a certain filter which is described explicitly. It is also possible to define explicitly a filter generating a class containing all Baire functions. Assuming the continuum hypothesis, it can be shown that the class of all Baire functions is generated e.g. by the intersection of two ultrafilters. Various other theorems concerning filters, their types and filter-generated classes are included, though some of them are not directly connected with problems of the descriptive classification.

No proofs are given. Some of the results, and also their proofs, are contained in the author's paper "On descriptive classes of functions" (referred to as DFC here), which is to appear in "Abhandlungen aus Mengenlehre und Topologie, dem Andenken Felix Hausdorffs gewidmet", Greifswald.

1. Preliminaries

1.1. We use the standard terminology and notation with slight modifications. The ordered pair x, y is denoted by $\langle x, y\rangle$. Symbols like $\{x_a \mid a \in A\}$, $\{x \mid f(x) < 0\}$, etc. denote either a set or a family (an indexed set); the meaning will always be clear from the context. If M is a set, we put $\exp M = \{X \mid X \subset M\}$.

"Space" will mean a topological space. As a rule, the same symbol will stand for a space and its underlying set.

1.2. Let a set A be given. Then, for any $\mathscr{M} \subset \exp A$, $\mathscr{M}^*$ denotes the collection of the sets $X \subset A$ intersecting all $M \in \mathscr{M}$.

1.3. If T is a set or a space, then $F(T)$ denotes the set of all real-valued functions on T, and 2^T denotes the set of all families $\{u_t \mid t \in T\}$ with $u_t = 0, 1$. As a rule, $F(T)$ and 2^T are considered as spaces with the topology of Cartesian product. The set of all continuous $f \in F(T)$ is denoted by $C(T)$. The canonical mapping of $\exp T$ onto 2^T is denoted by χ_T; if $\mathcal{M} \subset \exp T$, we usualy write $\chi\mathcal{M}$ instead of $\chi_T(\mathcal{M})$.

Remark. As usual, we consider functions on distinct spaces as distinct; thus, $F(T_1) \cap F(T_2) = \emptyset$ whenever T_1, T_2 are distinct spaces (possibly with the same underlying set).

1.4. A non-void collection $\mathcal{F} \subset \exp A$ is called a *quasi-filter* on A if (1) A is countably infinite, (2) $X \in \mathcal{F}$, $Y \in \mathcal{F}$, $X \cap Y \subset Z \subset A$ implies $Z \in \mathcal{F}$, (3) $\emptyset$ non $\in \mathcal{F}$. If, in addition, $\bigcap\mathcal{F} = \emptyset$, then $\mathcal{F}$ is called a *filter*. Thus, we consider filters on countably infinite sets only.

1.5. Convention. Whenever a definition refers to filters, it is tacitly assumed that it is also valid for quasi-filters.

Letters $\mathcal{F}$, $\mathcal{G}$, $\mathcal{U}$, $\mathcal{V}$, possibly with subscripts, will denote filters. The Fréchet filter on N, consisting of all $X \subset \mathsf{N}$ with $\mathsf{N} - X$ finite, will be denoted by $\mathcal{N}$.

1.6. If $\mathcal{F}$ is a filter on A, and $B \in \mathcal{F}^*$, then the collection of all $F \cap B$, $F \in \mathcal{F}$, is a filter on B, which will be called the *trace* of $\mathcal{F}$ on B.

1.7. If $\{X_a \mid a \in A\}$ is a family of sets, we denote by $\sum\{X_a \mid a \in A\}$ the set of all $\langle a, x\rangle$ where $a \in A$, $x \in X_a$. Let $\mathcal{F}$ be a filter on A; for every $a \in A$, let $\mathcal{G}_a$ be a filter on a set B_a. Then the collection of all $\sum\{G_a \mid a \in F\} \cup \sum\{H_a \mid a \in A - F\}$, where $F \in \mathcal{F}$, $G_a \in \mathcal{G}_a$, $H_a \subset B_a$, is a filter on $\sum\{B_a \mid a \in A\}$. It will be denoted by $\mathcal{F}$-$\sum\{\mathcal{G}_a \mid a \in A\}$.

1.8. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on A and B, respectively. Put $\mathcal{G}_a = \mathcal{G}$ for every $a \in A$. The filter $\mathcal{F}$-$\sum\{\mathcal{G}_a\}$ is denoted by $\mathcal{F} . \mathcal{G}$ and called the *product* of filters $\mathcal{F}$ and $\mathcal{G}$. The filter $\mathcal{F} . \mathcal{F}$ is denoted by $\mathcal{F}^2$.

1.9. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on A and B, respectively. Let φ be a single-valued relation with the domain A, ranging in B and such that $\varphi^{-1}(G) \in \mathcal{F}$ whenever $G \in \mathcal{G}$. The triple $\langle \varphi, \mathcal{F}, \mathcal{G}\rangle$ is called a morphism from $\mathcal{F}$ to $\mathcal{G}$. If $\langle \varphi^{-1}, \mathcal{G}, \mathcal{F}\rangle$ is also a morphism, then $\langle \varphi, \mathcal{F}, \mathcal{G}\rangle$ is called an isomorphism.

Convention. The terminology and notation used for mappings will be also applied to morphisms of filters.

1.10. If there exists a morphism from $\mathcal{F}$ to $\mathcal{G}$, we shall write $\mathcal{F} \geqq \mathcal{G}$. Clearly, $\geqq$ is a transitive reflexive relation on the class of all quasi-filters.

1.11. Proposition. *In the class of all quasi-filters endowed with the quasiorder $\geqq$, every countable non-void subset has a supremum and an infimum, and every subset of cardinality $\leqq \exp \aleph_0$ is bounded.*

1.12. Definition. If there exists an isomorphism from $\mathscr{F}$ to $\mathscr{G}$, we shall write $\mathscr{F} \cong \mathscr{G}$ and say that $\mathscr{F}$ is *isomorphic* to $\mathscr{G}$. If $\mathscr{F} \geqq \mathscr{G}$, $\mathscr{G} \geqq \mathscr{F}$, we shall write $\mathscr{F} \approx \mathscr{G}$ and say that $\mathscr{F}$ is *equivalent* to $\mathscr{G}$. Observe that "equivalent" has been used in [1] and in [2] in the sense of "isomorphic" as just defined.

1.13. Proposition. *Two equivalent ultrafilters are isomorphic.*

See e.g. [2], Proposition 1.15. – Observe that a filter equivalent to an ultrafilter need not be isomorphic to it.

2. Some properties of filters

2.1. If A is a set and $\boldsymbol{P}$ is a property of subsets of the space 2^T, then $\mathscr{M} \subset \exp A$ will be said to possess the property χ-$\boldsymbol{P}$, if $\chi\mathscr{M}$ (see 1.3) has property $\boldsymbol{P}$; the prefix "χ-" will be often omitted. E.g., $\mathscr{M} \subset \exp A$ is called Souslin if $\chi\mathscr{M}$ is a Souslin (= analytic) subset of 2^A.

2.2. Proposition. *A filter is Souslin if it has a base which is a Souslin collection.*

2.3. If A is a countably infinite set, then μ_A denotes the "canonical" measure on 2^A. If $\mathscr{M} \subset \exp A$, then the μ_A-measure of $\chi\mathscr{M}$ will be called the measure of $\mathscr{M}$; and similarly for other related concepts, in accordance with 2.1. – Cf. DCF, 4.3.

2.4. Theorem. *The interior measure of every filter is zero. No ultrafilter is measurable. The intersection of a non-measurable filter (on a set A) and an ultrafilter (on the same set) is not measurable.* – See DCF, 4.4.

3. Types of filters. Filter-limits

3.1. Definition. Two single-valued relations, typ and Typ, are chosen once for all. The domain of both typ and Typ is the class of all quasi-filters. The equality typ $\mathscr{F} =$ $=$ typ $\mathscr{G}$ holds if and only if $\mathscr{F}$ is isomorphic to $\mathscr{G}$; Typ $\mathscr{F}$ = Typ $\mathscr{G}$ holds if and only if $\mathscr{F}$ is equivalent to $\mathscr{G}$. We shall call typ $\mathscr{F}$ and Typ $\mathscr{F}$ the *isomorphism type* and the *equivalence type* of $\mathscr{F}$, respectively. We put typ $\mathscr{F} \geqq$ typ $\mathscr{G}$, Typ $\mathscr{F} \geqq$ $\geqq$ Typ $\mathscr{G}$ if and only if $\mathscr{F} \geqq \mathscr{G}$. If ξ, η are equivalence types, then $\xi > \eta$ means that $\xi \geqq \eta$ and $\xi \neq \eta$.

Observe that isomorphism types of filters are called simply "types of filters" in [2]. It is to be noted that isomorphism types (called simply "types") of ultrafilters have been introduced by Z. Frolík [1].

3.2. Clearly, all those quasi-filters which are not filters are equivalent; their equivalence type will be denoted by et 0 or simply by 0. The equivalence type of $\mathscr{N}$ be denoted by et 1 or simply by 1.

3.3. It is easy to show that the cardinality of the set of isomorphism types, and also that of the set of equivalence types is exp exp $\aleph_0$.

3.4. Theorem. *A countable non-void set of isomorphism types of filters has a supremum and an infimum. A countable set of equivalence types of filters has a unique supremum and a unique infimum.* — See 1.11.

3.5. Theorem. *If M is a set of cardinality $\leqq$ exp $\aleph_0$ consisting of isomorphism types (or equivalence types) of filters, then M is bounded.* — See 1.11.

3.6. Let P be a Hausdorff space. If $\mathscr{F}$ is a quasi-filter on a set A and $\{x_a \mid a \in A\}$ is a family of points of P, then the *$\mathscr{F}$-limit* of $\{x_a\}$, denoted by $\mathscr{F}$-lim $\{x_a\}$, is defined in the usual way: $x = \mathscr{F}\text{-lim}\,\{x_a\}$, if, for any neighborhood V of x, there is a set $F \in \mathscr{F}$ such that $x_a \in V$ whenever $a \in F$.

3.7. If P is a Hausdorff space and $\mathscr{F}$ is a filter, then, for any $S \subset P$, the set of all points $\mathscr{F}$-lim $\{x_a\}$, $x_a \in S$, will be denoted by $\mathscr{F}$-Lim S. If P and S are fixed, we shall sometimes say that $\mathscr{F}$ *generates* the set $\mathscr{F}$-lim S (and also every point $y \in \mathscr{F}$-Lim S).

3.8. If T is a set, $\mathscr{F}$ is a filter on a set A, and $\{f_a \mid a \in A\}$ is a family of functions on T, then the *upper* and the *lower $\mathscr{F}$-limit* of $\{f_a\}$ are defined in the usual way. E.g., the value assumed by the upper $\mathscr{F}$-limit of $\{f_a\}$ at a point $t \in T$ is equal to the g.l.b. of numbers ξ such that $\{a \mid f_a(t) < \xi\}$ belongs to $\mathscr{F}$; values $-\infty$ and ∞ are admitted. The upper and the lower $\mathscr{F}$-limit of $\{f_a\}$ will be denoted by $\mathscr{F}\text{-lim}^+ \{f_a\}$ and $\mathscr{F}\text{-lim}^- \{f_a\}$ respectively.

3.9. If T is a set or a space, $\mathscr{F}$ is a filter (on a set A) and $Y \subset F(T)$, then we denote by $\mathscr{F}\text{-Lim}^+ Y$ the set of all $f \in F(T)$ such that $f = \mathscr{F}\text{-lim}^+ \{f_a\}$ for some $f_a \in Y$. The symbol $\mathscr{F}\text{-Lim}^- Y$ is defined in an analogous way.

3.10. Proposition. *Let P be a Hausdorff space, $S \subset P$. If $\mathscr{F} \geqq \mathscr{G}$, then $\mathscr{F}$-Lim $S \supset \mathscr{G}$-Lim S. If $\mathscr{F} \approx \mathscr{G}$, then $\mathscr{F}$-Lim $S = \mathscr{G}$-Lim S.*

3.11. If P is a Hausdorff space, $S \subset P$, and ξ is an equivalence type, by 3.10, we may put ξ-Lim $S = \mathscr{F}$-Lim S, where Typ $\mathscr{F} = \xi$. If P and S are fixed, we shall say that ξ *generates* the set ξ-Lim S (and also every point $y \in \xi$-Lim S).

3.12. Theorem. *If P is a space admitting a one-to-one continuous mapping onto a separable metrizable space, then every $f \in F(T)$ is in some $\mathscr{F}$-Lim $C(T)$.*

3.13. Remarks. 1) $\mathscr{F} \geqq \mathscr{G}$ does not imply $\mathscr{F}\text{-Lim}^+ C(T) \supset \mathscr{G}\text{-Lim}^+ C(T)$. Example: $\mathscr{F}$, $\mathscr{U}$ are non-isomorphic ultrafilters on N, $\mathscr{G} = \mathscr{F} \cap \mathscr{U}$. 2) $\mathscr{F} \approx \mathscr{G}$ does not imply $\mathscr{F}\text{-Lim}^+ C(T) = \mathscr{G}\text{-Lim}^+ C(T)$.

3.14. Problems. 1) Does there exist a Hausdorff space P such that $\mathscr{F} \geqq \mathscr{G}$ whenever $\mathscr{F}$-Lim $S \supset \mathscr{G}$-Lim S for every $S \subset P$? 2) Does there exist a Hausdorff space P such that $\mathscr{F} \approx \mathscr{G}$ whenever $\mathscr{F}$-Lim $S = \mathscr{G}$-Lim S for every $S \subset P$?

4. Filter-descriptive classes and types of functions

4.1. We denote by $\mathscr{T}$ the class of all topological spaces. For any filter $\mathscr{F}$, we put $\mathrm{Cl}(\mathscr{F}) = \bigcup\{\mathscr{F}\text{-Lim}\, C(T) \mid T \in \mathscr{T}\}$, $\mathrm{Cl}^+(\mathscr{F}) = \bigcup\{\mathscr{F}\text{-Lim}^+ C(T) \mid T \in \mathscr{T}\}$, $\mathrm{Cl}^-(\mathscr{F}) = \bigcup\{\mathscr{F}\text{-Lim}^- C(T) \mid T \in \mathscr{T}\}$. If ξ is an equivalence type, we may, by 3.11, put $\mathrm{Cl}(\xi) = \mathrm{Cl}(\mathscr{F})$ where $\mathrm{Typ}\,\mathscr{F} = \xi$. However, it is not possible to define $\mathrm{Cl}^+(\xi)$ in an analogous way; see 3.13.

We shall call $\mathrm{Cl}(\mathscr{F})$ the *bilateral filter-descriptive class* (*descriptive class*, for short) *generated* by $\mathscr{F}$ (or by $\mathrm{Typ}\,\mathscr{F}$); $\mathrm{Cl}^+(\mathscr{F})$ and $\mathrm{Cl}^-(\mathscr{F})$ will be called the *upper* and the *lower unilateral descriptive class* (abbreviated: *upper* and *lower descriptive class*) *generated by* $\mathscr{F}$.

Examples: $\mathrm{Cl}(0)$ consists of all continuous functions on topological spaces; $\mathrm{Cl}(1) = \mathrm{Cl}(\mathscr{N})$ consists of all functions of the first Baire class.

4.2. Proposition. *If* $\mathscr{F} \geqq \mathscr{G}$, *then* $\mathrm{Cl}(\mathscr{F}) \supset \mathrm{Cl}(\mathscr{G})$. *If* $\mathscr{F} \approx \mathscr{G}$, *then* $\mathrm{Cl}(\mathscr{F}) = \mathrm{Cl}(\mathscr{G})$. – See 3.10.

4.3. Theorem. *If T is a space admitting a one-to-one continuous mapping onto a separable metrizable space, then every real-valued function on T is in some filter-descriptive class.* – See 3.12.

4.4. Proposition. *For every $n \in \mathrm{N}$, let $\mathscr{D}_n$ be a bilateral filter-descriptive class of functions. Then $\mathscr{D} = \{f \mid f \in \mathscr{D}_n$ for every $n \in \mathrm{N}\}$ is a bilateral filter-descriptive class; if $\mathscr{D}_n = \mathrm{Cl}(\mathscr{F}_n)$, then $\mathscr{D}$ is generated by an infimum of filters $\mathscr{F}_n$ (or: by the infimum of types* $\mathrm{Typ}\,\mathscr{F}_n$). – See 1.11.

4.5. Proposition. *For every $n \in \mathrm{N}$, let $\mathscr{D}_n$ be a bilateral filter-descriptive class of functions. Then there exists a smallest bilateral filter-descriptive class containing all $\mathscr{D}_n$. If $\mathscr{D}_n = \mathrm{Cl}(\mathscr{F}_n)$, then this class is generated by a supremum of filters $\mathscr{F}_n$.* – See 1.11.

4.6. Proposition. *Let M be a set of cardinality $\leqq \exp \aleph_0$. For every $m \in M$, let $\mathscr{C}_m$ be a bilateral filter-descriptive class. Then there exists a filter-descriptive class $\mathscr{C}$ such that $\mathscr{C}_m \subset \mathscr{C}$ for every $m \in M$.* – See 1.11.

4.7. Let T be a non-void compact metrizable space. Denote by $\mathscr{Y}(T)$ the collection of all countable $Y \subset C(T)$ which are dense in $C(T)$ endowed with the topology of uniform convergence. If $f \in F(T)$, $Y \in \mathscr{Y}(T)$, put $Y_f = Y \cup (f)$ if f is continuous, $Y_f = Y$ if not. Denote by $\Phi(f, Y)$ the filter on Y_f a subbase of which consists of all sets $\{g \mid g \in Y_f,\ f(t) - \varepsilon < g(t) < f(t) + \varepsilon\}$, where $t \in T$, $\varepsilon > 0$. It is easy to show that, for any Y_1, Y_2 in $\mathscr{Y}(T)$, $\Phi(f, Y_1)$ is isomorphic to $\Phi(f, Y_2)$. We put $\Phi(f) = \operatorname{Typ} \Phi(f, Y)$ where $Y \in \mathscr{Y}(T)$; $\Phi(f)$ will be called the *descriptive type* of f.

4.8. Theorem. *Let T be a non-void compact metrizable space. If $\mathscr{F}$ is a quasi-filter, then a function $f \in F(T)$ belongs to* $\mathrm{Cl}(\mathscr{F})$ *if and only if* $\operatorname{Typ} \mathscr{F} \geqq \Phi(f)$.

In other words, the descriptive type of a function $f \in F(T)$ is the infimum (in fact, the "minimum") of equivalence types of filters generating f.

4.9. Problems. 1) Does $\mathrm{Cl}^+(\mathscr{F}) = \mathrm{Cl}^+(\mathscr{G})$ imply $\mathscr{F} \approx \mathscr{G}$ (or even $\mathscr{F} \cong \mathscr{G}$)? 2) Does $\mathrm{Cl}(\mathscr{F}) = \mathrm{Cl}(\mathscr{G})$ imply $\mathscr{F} \approx \mathscr{G}$? 3) Does $\mathrm{Cl}(\mathscr{F}) \supset \mathrm{Cl}(\mathscr{G})$ imply $\mathscr{F} \geqq \mathscr{G}$? 4) Is there a compact metrizable T, a function $f \in F(T)$ and an ultrafilter $\mathscr{U}$ such that $\Phi(f) \geqq \operatorname{Typ} \mathscr{U}$? 5) If T is a topological space, does there always exist an infimum of all equivalence types of filters generating f? If such an infimum exists, does it generate the function f?

5. Special filters and transforms of filters

5.1. Definition. We put $\mathscr{N}^0 = \{X \mid X \subset \mathsf{N},\ 0 \in X\}$, $D^{(0)} = \mathsf{N}$, $\mathscr{N}^1 = \mathscr{N}$, $D^{(1)} = \mathsf{N}$. Let $\alpha > 1$ be a countable ordinal and suppose that the filters $\mathscr{N}^\xi$ on $D^{(\xi)}$ have been defined for all $\xi < \alpha$. If $\alpha = \beta + 1$, we put $\mathscr{N}^\alpha = \mathscr{N} . \mathscr{N}^\beta$, $D^{(\alpha)} = \mathsf{N} \times D^{(\beta)}$. If α is a limit ordinal, we put $\mathscr{N}^\alpha = \mathscr{O}(\alpha)\text{-}\sum\{\mathscr{N}^\xi \mid 0 < \xi < \alpha\}$ where $\mathscr{O}(\alpha)$ denotes the filter on $\{\xi \mid 0 < \xi < \alpha\}$ with a base consisting of all sets $\{\xi \mid \beta < \xi < \alpha\}$, $\beta < \alpha$. Then $\mathscr{N}^\alpha$ is a filter on the set $D^{(\alpha)} = \sum\{D^{(\xi)} \mid 0 < \xi < \alpha\}$. – See DCF, 2.7.

5.2. The equivalence type of $\mathscr{N}^\alpha$ will be denoted by et α or simply by α.

5.3. Proposition. *If α, β are countable ordinals, then $\mathscr{N}^\alpha . \mathscr{N}^\beta$ is isomorphic to $\mathscr{N}^{\beta+\alpha}$.* – See DCF, 2.8.

5.4. Definition. If A is a set, then $\mathrm{e}A$ will denote the collection of all finite subsets of A. If $\mathscr{F}$ is a filter on A, consider the collection of sets $\{x \mid x \in \mathrm{e}A,\ x \subset F\}$ where $F \in \mathscr{F}$, and $\{x \mid x \in \mathrm{e}A,\ x \cap H \neq \emptyset\}$ where $H \in \mathscr{F}^*$ (see 1.2). Clearly, this collection is a subbase of a filter on $\mathrm{e}A$, which will be denoted by $\mathbf{e}\mathscr{F}$ and called the **e**-transform of $\mathscr{F}$.

5.5. Proposition. *For any filter $\mathscr{F}$, $\mathbf{e}\mathscr{F} \geqq \mathscr{F}$.*

5.6. Proposition. *If $\mathscr{U}$ is an ultrafilter, then $\mathbf{e}\mathscr{U} \approx \mathscr{U}$.*

In more detail: $\mathscr{U}$ is isomorphic to the trace of $\mathbf{e}\mathscr{U}$ on the set $\{(x) \mid x \in A\} \subset \mathrm{e}A$.

5.7. Proposition. $\mathcal{N} \leqq \mathbf{e}\mathcal{N} \leqq \mathcal{N}^2$.

5.8. If A is a set, then $\mathrm{w}A$ will denote the set of all finite sequences of elements of A. If $\alpha \in \mathrm{w}A$, $\beta \in \mathrm{w}A$, then $\alpha \,.\, \beta$ denotes the "concatenation" of α and β (i.e., if $\alpha = \{a_0, \ldots, a_m\}$, $\beta = \{b_0, \ldots, b_n\}$, then $\alpha \,.\, \beta = \{a_0, \ldots, a_m, b_0, \ldots, b_n\}$).

5.9. Definition. A set $V \subset \mathrm{w}A$ will be called *sequentially finite*, if, for any $\alpha \in A^{\mathsf{N}}$, there are only finitely many such segments of α which are also segments of some $\beta \in V$.

5.10. Lemma. *If A is countable non-void, then the collection of complements of sequentially finite sets is a filter on* $\mathrm{w}A$.

5.11. Let $\mathcal{F}$ be a filter on A. For any $\varphi \in \mathcal{F}^{\mathrm{w}A}$, we denote by $G(\varphi)$ the set of all $\alpha \in \mathrm{w}A$ possessing the following property: if $\alpha = \beta \,.\, \{c\} \,.\, \delta$, then $c \in \varphi(\beta)$. It is easy to show that, for any $\varphi_i \in \mathcal{F}^{\mathrm{w}A}$, $G(\varphi_1) \cap G(\varphi_2) = G(\varphi)$ where $\varphi \in \mathcal{F}^{\mathrm{w}A}$, $\varphi(\alpha) = \varphi_1(\alpha) \cap \varphi_2(\alpha)$ for every $\alpha \in \mathrm{w}A$, and that no $G(\varphi)$ is sequentially finite.

We denote by $\mathbf{w}\mathcal{F}$ the collection of all $H \subset \mathrm{w}A$ such that $H \supset G(\varphi) - V$ for some $\varphi \in \mathcal{F}^{\mathrm{w}A}$ and some sequentially finite V. It is easily seen that $\mathbf{w}\mathcal{F}$, called the $\mathbf{w}$-transform of $\mathcal{F}$, is a filter on $\mathrm{w}A$.

5.12. Proposition. *For any filter $\mathcal{F}$, $(\mathbf{w}\mathcal{F})^2 \geqq \mathbf{w}\mathcal{F} \geqq \mathcal{F}$.*

5.13. Theorem. *For any filter $\mathcal{F}$, both $\mathcal{F} \,.\, \mathbf{w}\mathcal{F}$ and $(\mathbf{w}\mathcal{F}) \,.\, \mathcal{F}$ are isomorphic to $\mathbf{w}\mathcal{F}$.*

5.14. Theorem. *For any filter $\mathcal{F}$, the class* $\mathrm{Cl}\,(\mathbf{w}\mathcal{F})$ *is closed under $\mathcal{F}$-limits (hence also under $\mathcal{N}$-limits, i.e. the usual limits of sequences).*

5.15. Let T be a space. A set $S \subset T$ will be called a *Souslin set* (in T) if it can be obtained by the Souslin operation from a family $\{X_\alpha \mid \alpha \in \mathrm{w}\mathsf{N}\}$ where X_α are zero-sets in T, i.e. sets of the form $\{t \mid t \in T, f(t) = 0\}$, $f \in C(T)$. A function $f \in F(T)$ will be called *Souslin* (*co-Souslin*) if all sets $\{t \mid t \in T, f(t) > c\}$, c a real number, are Souslin (co-Souslin).

5.16. Theorem. *The class* $\mathrm{Cl}\,(\mathbf{w}\mathcal{N})$ *contains all Souslin and all co-Souslin functions.*

5.17. Remark. It can be shown that $\chi(\mathbf{w}\mathcal{N}) \subset 2^{\mathrm{w}\mathsf{N}}$ belongs to the projective class ***PCA*** (for the definition of this class see e.g. [3]).

5.18. Problems. 1) Does there exists an equivalence type ν such that $1 < \nu < 2$? In a more general way, do there exist, for countable ordinals ξ, equivalence types ν_ξ

with $\xi < \nu_\xi < \xi + 1$? 2) Is there a filter $\mathscr{G}$ with $\mathscr{G} \approx \mathscr{G}^2$? Does $(\mathbf{w}\mathscr{F})^2 \approx \mathbf{w}\mathscr{F}$ hold for some $\mathscr{F}$? For $\mathscr{F} = \mathscr{N}$? For all $\mathscr{F}$? 3) The same questions with $\cong$ instead of $\approx$. 4) Is $\mathbf{w}\mathscr{N}$ measurable (see 2.3)?

6. Unilateral and bilateral descriptive classes

6.1. Theorem. *Let T be a set. Let $D \subset F(T)$ possess the following property: if $f_1 \in D$, $f_2 \in D$, $\varepsilon > 0$, then there is a function $g \in D$ such that $|g(t) - \max(f_1(t), f_2(t)| < \varepsilon$ for all $t \in T$. Then, for any filter $\mathscr{F}$, $\mathscr{F}\text{-Lim}^+ D \subset \mathbf{e}\mathscr{F}\text{-Lim}\, D$, $\mathscr{F}\text{-Lim}^- D \subset \subset \mathbf{e}\mathscr{F}\text{-Lim}\, D$.*

6.2. Theorem. *Let T be a topological space. Then, for any filter $\mathscr{F}$, we have $\mathrm{Cl}^+(\mathscr{F}) \subset \mathrm{Cl}(\mathbf{e}\mathscr{F})$, $\mathrm{Cl}^-(\mathscr{F}) \subset \mathrm{Cl}(\mathbf{e}\mathscr{F})$. Thus, every unilateral filter-descriptive class is contained in a bilateral filter-descriptive class.*

7. Baire functions and filter-descriptive classes

7.1. Theorem. *For every countable ordinal α, the filter-descriptive class $\mathrm{Cl}(\mathrm{et}\,\alpha)$ consists of all functions (on topological spaces) of Baire class α.* – See DCF, 2.17.

7.2. Theorem. *Under the continuum hypothesis, there exist ultrafilters $\mathscr{U}$, $\mathscr{V}$ on N such that $\mathscr{U} \cap \mathscr{V}$ generates the class of all Baire functions on topological spaces.* – See DCF, 5.6.

7.3. Theorem. *Under the continuum hypothesis, there exists an ultrafilter $\mathscr{U}$ (on $\mathrm{w}\mathsf{N}$) such that $\mathscr{U} \cap \mathbf{w}\mathscr{N}$ generates the class of all Baire functions on topological spaces.*

Added in proof. The author wishes to point out that filtered sums of filters (see 1.7) and products of filters (see 1.8) have been introduced by G. Grimeisen [4]. A part of Theorem 7.1 is due, in a different setting, to G. Grimeisen [5].

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