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TOPOLOGICAL SPACES WHICH ADMIT A COMPATIBLE COMPLETE QUASI-UNIFORMITY

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1. Introduction

The concept of a quasi-uniformity on a set X was introduced by L. Nachbin [13]. A *quasi-uniformity on a set* X is a filter $\mathscr{U}$ on $X \times X$ with the following two properties

i) if $U \in \mathscr{U}$, then $\Delta = \{(x, x) : x \in X\} \subset U$

ii) if $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $V \circ V \subset U$.

If $\mathscr{U}$ is a quasi-uniformity on a set X, then $\mathscr{T}_{\mathscr{U}} = \{A \subset X : \text{if } x \in A, \text{ there is } V \in \mathscr{U} \text{ with } V(x) \subset A\}$ is a topology for X. If X is a set, $\mathscr{U}$ is a quasi-uniformity on X and $\mathscr{T}$ is a topology on X, then $\mathscr{U}$ is *compatible* with $(X, \mathscr{T})$ provided that $\mathscr{T} = \mathscr{T}_{\mathscr{U}}$. It is known that every topological space admits a compatible quasi-uniformity. In this paper we consider the question, does every topological space admit a compatible complete quasi-uniformity.

Throughout this paper, if $\mathscr{C}$ is an open cover of a topological space $(X, \mathscr{T})$ and $x \in X$ then $A_x^{\mathscr{C}}$ denotes $\bigcap\{C \in \mathscr{C} : x \in C\}$.

2. Compatible quasi-uniformities

In 1955, V. S. Krishnan proved essentially the following result.

Theorem 2.1 [12]. *Let $(X, \mathscr{T})$ be a topological space and let U be the collection of all upper semi-continuous functions on $(X, \mathscr{T})$. For each $\varepsilon > 0$ and each $f \in U$, let $U_{(f,\varepsilon)} = \{(x, y) \in X \times X : f(y) - f(x) < \varepsilon\}$. The quasi-uniformity $\mathscr{U}$ generated by $\{U_{(f,\varepsilon)} : f \in U, \varepsilon > 0\}$ is compatible with $(X, \mathscr{T})$.*

The quasi-uniformity of Theorem 2.1 is called the upper semi-continuous quasi-uniformity and is denoted by $\mathscr{U}\mathscr{S}\mathscr{C}$. A *Q-cover* of a topological space $(X, \mathscr{T})$ is an open cover $\mathscr{C}$ of X such that if $x \in X$, then $A_x^{\mathscr{C}} = \bigcap\{C \in \mathscr{C} : x \in C\} \in \mathscr{T}$ [15].

Theorem 2.2. [6]. *Let $\mathscr{A}$ be the collection of all Q-covers of a topological space $(X, \mathscr{T})$. For each $\mathscr{C} \in \mathscr{A}$,* let *$U_{\mathscr{C}} = \bigcup\{\{x\} \times A_x^{\mathscr{C}} : x \in X\}$ and let $\mathscr{B} = \{U_{\mathscr{C}} : \mathscr{C} \in \mathscr{A}\}$. Then $\mathscr{B}$ is a base for a compatible quasi-uniformity $\mathscr{F}\mathscr{T}$ for $(X, \mathscr{T})$.*

A quasi-uniformity $\mathscr{U}$ is *transitive* provided that there is a base $\mathscr{B}$ for $\mathscr{U}$ with the property that if $B \in \mathscr{B}$, then $B \circ B = B$. The quasi-uniformity of Theorem 2.2 is denoted by $\mathscr{F}\mathscr{T}$ because it is always the finest compatible transitive quasi-uniformity for a given space $(X, \mathscr{T})$ [6, Corollary to Theorem 2]. The authors have been unable to find an example of a space for which $\mathscr{F}\mathscr{T}$ is not in fact the fine quasi-uniformity. Consequently, it may well be that the problem discussed herein may reduce to the problem of deciding for which spaces $\mathscr{F}\mathscr{T}$ is complete.

3. Completeness

Definition [14]. Let $(X, \mathscr{U})$ be a quasi-uniform space. A filter $\mathscr{F}$ on X is *$\mathscr{U}$-Cauchy* provided that if $U \in \mathscr{U}$, there is $p \in X$ such that $U(p) \in \mathscr{F}$. The quasi-uniform space $(X, \mathscr{U})$ is *complete* provided that every $\mathscr{U}$-Cauchy filter converges.

It is natural to ask if there exists a non-realcompact Tychonoff space which admits a compatible complete quasi-uniformity. This question was answered by J. Carlson and T. Hicks who exhibited a compatible complete quasi-uniformity for $(\mathcal{O}, \Omega)$, the space of ordinals less than the first uncountable ordinal [3]. Actually, as we now show, it follows from known results that any linearly ordered space admits a compatible complete quasi-uniformity.

Definition [15]. A topological space $(X, \mathscr{T})$ is *orthocompact* (has property Q in terms of [15]) provided that every open cover has an open Q-refinement.

Definition [5]. A cover $\mathscr{R}$ of a space $(X, \mathscr{T})$ is a *fundamental* cover provided that if $x \in X$, then there is $U \in \mathscr{T}$ such that $x \in U \subset A_x^{\mathscr{R}}$. A topological space *has property F* provided that every open cover has a fundamental open refinement.

We note that in the literature the concept of orthocompactness [15] predates the concept of having property F [5].

Lemma. *A topological space is orthocompact if and only if it possesses property F.*

Proof. Let $\mathscr{C}$ be an open cover of X and let $\mathscr{R}$ be a fundamental open refinement. For each $x \in X$, let $U_x = \operatorname{int}(A_x^{\mathscr{R}})$ and let $\mathscr{S} = \{U_x : x \in X\}$. It is clear that $\mathscr{S}$ is an open refinement of $\mathscr{C}$. We now show that $\mathscr{S}$ is a Q-cover. Let $y \in X$ and let $p \in A_y^{\mathscr{S}}$. Let $q \in U_p$ and let $x \in X$ such that $y \in U_x$. Then $p \in U_x \subset A_x^{\mathscr{R}}$ and $q \in U_p \subset A_p^{\mathscr{R}}$ so that $q \in A_x^{\mathscr{R}}$. Thus $U_p \subset A_x^{\mathscr{R}}$. Since U_p is open, $U_p \subset \operatorname{int}(A_x^{\mathscr{R}}) = U_x$. Thus $U_p \subset U_x$ for each x such that $y \in U_x$. That is $U_p \subset A_y^{\mathscr{S}}$.

Lemma [6, Theorem 3]. *Every orthocompact topological space has a compatible complete transitive quasi-uniformity.*

Lemma [5, Theorem 3]. *Every linearly ordered topological space has property F.*

Theorem 3.1. *Every linearly ordered topological space has a compatible complete transitive quasi-uniformity.*

Definition [14]. A quasi-uniform space $(X, \mathscr{U})$ is *precompact* provided that if $U \in \mathscr{U}$, there is a finite subset F of X such that $U(F) = X$.

The classic proof of J. Dieudonné that $(\mathcal{O}, \Omega)$ does not possess a compatible complete uniformity does not depend upon Shirota's theorem; rather it makes use of a generalization of the Niemytski-Tychonoff theorem. A principal justification for defining completeness with respect to quasi-uniform spaces is that this same generalization still obtains for quasi-uniform spaces.

Theorem 3.2 [14, Theorem 2.2]. *Let $(X, \mathscr{U})$ be a quasi-uniform space. Then $(X, \mathscr{T}_{\mathscr{U}})$ is compact if and only if $\mathscr{U}$ is complete and precompact.*

We recall the argument of J. Dieudonné: One first shows that $(\mathcal{O}, \Omega)$ has only one compatible uniformity which must be precompact. By the above theorem, this uniformity cannot be complete since $(\mathcal{O}, \Omega)$ is not compact. This argument suggests the searching out of a non-compact topological space for which every compatible quasi-uniformity is precompact.

Theorem 3.3. *A topological space $(X, \mathscr{T})$ is compact if and only if $\mathscr{F}\mathscr{T}$ is precompact.*

Proof. By [7, Proposition 4.2], a necessary and sufficient condition that $\mathscr{F}\mathscr{T}$ be precompact is that every Q-cover have a finite subcover. Evidently every well-ordered (by set inclusion) open cover is a Q-cover, and it is known that a space is compact if and only if every well-ordered open cover has a finite subcover [1, Theorem 7.1].

In light of the above theorem we now seek a topological space whose fine quasi-uniformity is neither complete nor precompact.

Definition [9]. A quasi-uniform space $(X, \mathscr{U})$ is *almost precompact* provided that if $U \in \mathscr{U}$, there is a finite subset F of X such that $\overline{U(F)} = X$; and $(X, \mathscr{U})$ is *almost complete* provided that every open Cauchy filter has a cluster point.

Theorem 3.4 [2, Theorem 3.3]. *The upper semi-continuous quasi-uniformity $\mathscr{U}\mathscr{S}\mathscr{C}$ for a topological space $(X, \mathscr{T})$ is precompact if and only if $(X, \mathscr{T})$ is countably compact.*

Definition [10]. A filter $\mathscr{F}$ has the countable *subcollection intersection property* provided that if $\mathscr{C}$ is a countable subcollection of $\mathscr{F}$, then $\bigcap\{\overline{F} : F \in \mathscr{C}\} \neq \emptyset$. A space

$(X, \mathscr{T})$ is *almost realcompact* provided that every open ultrafilter with the countable subcollection intersection property converges.

The concept of almost realcompactness was defined in 1961 by Z. Frolík. Only recently has the following internal characterization of almost realcompactness been obtained.

Theorem 3.5 [7, Corollary to Theorem 5.1]. *A topological space is almost realcompact if and only if $\mathscr{U}\mathscr{S}\mathscr{C}$ is almost complete.*

Corollary. *Let $(X, \mathscr{T})$ be a topological space which is neither almost realcompact nor countably compact. Then $\mathscr{U}\mathscr{S}\mathscr{C}$ is neither complete nor precompact.*

Recall that a Tychonoff space is pseudocompact if and only if it is countably almost compact (=countably H-closed), and that very countably almost compact almost realcompact space is almost compact [10, Theorem 2].

Example. The space Ψ [11, page 79] is pseudocompact but not countably compact. Consequently $\mathscr{U}\mathscr{S}\mathscr{C}$ is neither complete nor precompact. It is easily seen that Ψ is orthocompact so that $\mathscr{F}\mathscr{T}$ is complete.

We do not know of an example of a normal almost precompact quasi-uniform space which is not precompact; however if $(X, \mathscr{T})$ is any pseudocompact Tychonoff space which is not countably compact, then $\mathscr{U}\mathscr{S}\mathscr{C}$ is an almost precompact quasi-uniformity, which is not precompact.

4. Questions

The authors believe that the following questions are related to the problem of deciding which topological spaces admit a compatible complete quasi-uniformity.

For which topological spaces is $\mathscr{F}\mathscr{T}$ the fine quasi-uniformity?

For which topological spaces is $\mathscr{U}\mathscr{S}\mathscr{C}$ the fine quasi-uniformity?

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