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A survey of the theory of generalized metric spaces

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A SURVEY OF THE THEORY OF GENERALIZED METRIC SPACES

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1. Introduction

In the present lecture a survey will be given on the remarkable development of theory of generalized metric spaces which has taken place mainly since 1966, the year of the second Prague Symposium. There are various surveys, [5], [51], [42], [57], [10], [25] on this aspect of general topology, so efforts will be made to avoid too much overlap with those works. (Note that all spaces in this lecture are at least T_1 though most definitions as well as some theorems may be valid for spaces without T_1; all paracompact spaces are Hausdorff, and all single-valued maps (=mappings) are continuous. As for general terminology and symbols in general topology, see [46].)

Our main concern are generalized metric spaces of the following two types.

Type 1. M, p and related spaces.
Type 2. M_i, σ and related spaces.

Some spaces discussed here may be classified to neither of the two types but only related to them to various extents. The importance of spaces of type 1 largely relies on the fact that they are general enough to generalize both metric spaces and compact spaces and still concrete enough to allow beautiful extensions of theorems from those classical spaces. The importance of spaces of type 2 relies on the fact that they have, in various aspects, nicer properties than metric spaces do. For example, if a space is the sum of countably many closed sets which are σ-spaces as subspaces, then it is a σ-space while the same is not true for metrizable spaces. Accordingly some popular spaces (e.g. some infinite complices with weak topology) are spaces of this type though they are not metrizable. Thus our knowledge of topological properties of complices, for example, is not satisfactory without a full study of type 2 spaces. (Paracompactness seems too general to well represent properties of many important spaces which are "nearly metrizable".)

2. Basic properties

Since M, p, σ and M_i $(i = 1, 2, 3)$-*spaces* are becoming quite popular (originated by K. Morita, A. Arhangelskii, A. Okuyama and J. Ceder, respectively), no definition of them will be given in this lecture. The reader is referred to [40], [5] and [10] for their definitions. The above basic concepts are accompanied by many generalizations and modifications which are less popular. M^* and $w\Delta$, for example, generalize M-space.

Definition 1 ([9], [29]). X is a *$w\Delta$- (M^*-) space* if it has a sequence $\{\mathscr{U}_i \mid i = 1, 2, \ldots\}$ of open covers (locally finite closed covers) satisfying

(M) if $x_i \in S(x, \mathscr{U}_i)$, $i = 1, 2, \ldots$ for a fixed point x, then the point sequence $\{x_i \mid i = 1, 2, \ldots\}$ has a cluster point.

M^*-space is especially interesting as a useful supplement to M-space because the former is often easier to handle than the latter while *M and M^* coincide for every normal space* as proved by T. Ishii [30]. Semi-stratifiable is an interesting concept to generalize stratifiable ($=M_3$-) space.

Definition 2 ([15]). X is a *semi-stratifiable space* if to every open set U of X we can assign a sequence $\{U_i \mid i = 1, 2, \ldots\}$ of closed sets such that

(i) $U = \bigcup_{i=1}^{\infty} U_i$,

(ii) if $U \subset V$ for open sets U and V, then $U_i \subset V_i$, $i = 1, 2, \ldots$.

When a new class $\mathscr{C}$ of spaces is established, we are interested in properties of $\mathscr{C}$ like those in the following.

(1) If $X \in \mathscr{C}$ and $X' \subset X$, then $X' \in \mathscr{C}$.

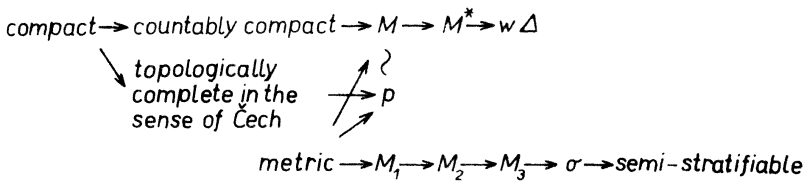
(2) If $X_i \in \mathscr{C}$, $i = 1, 2, \ldots$, then $\prod_{i=1}^{\infty} X_i \in \mathscr{C}$.

(3) If $X \in \mathscr{C}$, and there is a closed map from X onto Y, then $Y \in \mathscr{C}$.

(4) If $X = \bigcup_{i=1}^{\infty} X_i$ for closed sets $X_i \in \mathscr{C}$, $i = 1, 2, \ldots$, then $X \in \mathscr{C}$.

(5) If X is dominated by a closed cover $\{X_\alpha \mid \alpha \in A\}$ with $X_\alpha \in \mathscr{C}$, then $X \in \mathscr{C}$.

In this respect spaces of type 1 usually show poor quality. M-spaces, for example, satisfy none of (1)–(5); especially, as shown by T. Ishiwata [32], *the product of two Tychonoff M-spaces need not be M. The image of an M-space by a perfect map is not necessarily M* as proved by K. Morita [41]. Paracompact M-spaces satisfy only (2). (See [42].) On the contrary, as suggested in the previous section, spaces of type 2 usually show good records. σ-spaces, for example, have all of the five properties though regularity for (3) and normality for (5) are required (see [57]). Stratifiable ($=M_3$-) spaces have all but (4) (see [10]). As for (4), R. Heath [24] proved that *there is a regular space with countably many points which is not stratifiable.* Relations between the classes of generalized metric spaces and some classical spaces are partially given in the following diagram.



(We need Hausdorff axiom for the implication compact → topologically complete; M and p coincide if the space is paracompact though they differ in general case.) Most of the relations are easy to prove, but the implication $M_3 \to \sigma$ had been a major open question until *R. Heath* [23] *recently proved it.* On the other hand "$M_3 \to M_2$? $M_2 \to M_1$?" still remains as a major open question as it was posed by J. Ceder ten years ago. C. Borges recently proved in an unpublished paper that if each point of every stratifiable space has a σ-closure preserving nbd (local) base, then M_3 is M_1. But it is unknown if every stratifiable space has such nbd basis. In this connection we ask: *Is the image of a metric space by a closed map M_1 (or M_2)?*, which might be a little easier than Ceder's problem if we could expect affirmative answers at all. (Note that the closed image of a metric space is stratifiable. It has been announced quite recently that this problem has been solved affirmatively by F. Slaughter.) There are several remarkable relations which do not appear in the above diagram. For example,

Theorem 1 (G. Creede [15]). *X is semi-metric iff (=if and only if) it is first-countable and semi-stratifiable.*

3. Characterizations

There are various characterizations obtained of generalized metric spaces,which help us to study their different aspects and often lead us to further results. As for M-space the following is fundamental.

Theorem 2 (Morita [40]). *X is an M-space iff there is a quasi-perfect map f from X onto a metric space Y.*

The following theorems show how deeply paracompact M-spaces, which are especially interesting among M-spaces, are related to metric spaces and compact spaces.

Theorem 3 (Morita [40], Arhangelskii [2]). *The following conditions are equivalent.*

(i) *X is paracompact M,*

(ii) *X is paracompact p,*

(iii) *there is a perfect map f from X onto a metric space Y.*

Theorem 4 (V. Kljušin [34], Morita [43], P. Zenor). *X is paracompact M iff it is the limit of an inverse system of metric spaces with perfect bonding maps.*

Theorem 5 (J. Nagata [50]). *X is paracompact M iff it is homeomorphic to a closed set of the product of a metric space and a compact Hausdorff space.*

The last theorem leads us to the following two questions:

1. *Find out a universal space Z such that X is a paracompact M-space with weight α iff it is homeomorphic to a closed set of Z.* (It is also worth while to find universal spaces for the other classes of generalized metric spaces including those of type 2.)

2. *Is every M-space homeomorphic to a closed set of the product of a metric space and a countably compact space*? (The converse is easily seen to be true.) A special case of the first problem is answered as follows.

Theorem 6 (Nagata [50]). *X is a paracompact, topologically complete* (in the sense of Čech) *space with weight α iff it is homeomorphic to a closed set of $H(A) \times P(A)$, where $H(A)$ and $P(A)$ are the Hilbert space with α coordinates and the product of α closed intervals, respectively.*

As well-known p-space and a little stronger concept, *strict p-space* are defined in terms of open families in Stone-Čech compactification, and this fact prevents us from handling them with more easiness or comparing them with M-space. D. Burke [12] gave internal characterizations of p- and strict p-spaces and studied their relations with $w\Delta$-spaces.

Theorem 7. *A Tychonoff space X is p iff there is a sequence $\{\mathcal{U}_i\}$ of open covers of X satisfying: If $x \in X$ and $x \in U_i \in \mathcal{U}_i$, $i = 1, 2, \ldots$, then*

(i) $\bigcap_{i=1}^{\infty} \overline{U}_i$ *is compact,*

(ii) *if* $x_n \in \bigcap_{i=1}^{n} \overline{U}_i$, $n = 1, 2, \ldots$, *then* $\{x_n\}$ *has a cluster point.*

Turning to spaces of type 2, the following characterization is basic for σ-spaces, which indicates that the natures of "base" and "*net* (or *network*)" are essentially different despite their similar definitions, and this fact contributes to the advantageous properties of σ-spaces in comparison with metric spaces.

Theorem 8 (F. Siwiec and J. Nagata [62]). *The following conditions for a regular space X are equivalent.*

(i) *X is σ,*

(ii) *X has a σ-closure preserving net,*

(iii) *X has a σ-discrete net.*

Since we have seen so many generalized metric spaces emerging up one after another, it is natural to try to characterize them in a unified manner. One of such attempts is done by R. Heath, R. Hodel and others by use of open nbds (= neighborhoods). Let $\{U(n, x) \mid n = 1, 2, \ldots\}$ be a sequence of open nbds of $x \in X$; then we consider e.g. the following conditions.

(1) $\{U(n, x) \mid n = 1, 2, \ldots\}$ is a nbd base for x.

(2) If $y \in U(n, x)$, then $U(n, y) \subset U(n, x)$.

(3) If $x \notin F$ for a closed set F, then $x \notin \bigcup\{U(n, y) \mid y \in F\}$ for some n. (This is equal to: If $x \in U(n, x_n)$, $n = 1, 2, \ldots$, then x is a cluster point of $\{x_n\}$.)

(4) If $x \notin F$ for a closed set F, then $x \notin [\bigcup\{U(n, y) \mid y \in F\}]^-$ for some n.

(5) If $\{x, x_n\} \subset U(n, y_n)$, $n = 1, 2, \ldots$, then x is a cluster point of $\{x_n\}$.

(6) If $\{x, x_n\} \subset U(n, y_n)$, $n = 1, 2, \ldots$, then $\{x_n\}$ has a cluster point.

(7) If $U(n, x) \cap U(n, x_n) \neq \emptyset$, $n = 1, 2, \ldots$, then x is a cluster point of $\{x_n\}$.

Theorem 9. *Generalized metric spaces are characterized in terms of existence of $\{U(n, x)\}$ satisfying the above conditions as follows: Semi-stratifiable* = (3) (Creede [16]), *semi-metric* = (1) *and* (3) (Heath [20]), σ = (2) *and* (3) (Heath-Hodel [26]), *stratifiable* = (4) (Heath [21]), *Nagata space* (= *stratifiable and first countable*) = (7) (Heath [21]), M_2 = (2) *and* (4) (Nagata [54]), *developable* = = (5) (Heath [20]), $w\Delta$ = (6) (Heath [20]). (Developable spaces are related to spaces of both type 1 and type 2, because every developable space is semi-metric and σ, and every Tychonoff developable space is strict p.)

A merit of this characterization is to help us to better understand relations between different spaces. In fact (4) was used by Heath to prove the implication stratifiable $\rightarrow \sigma$, and Hodel used (5) and (6) to study the relation between developable spaces and $w\Delta$-spaces. Ceder's question, $M_3 \rightarrow M_2$? is restated as follows: Given nbds satisfying (4) and nbds satisfying (2) and (3), is it then possible to construct nbds satisfying (4) and (2) at the same time?

4. Mappings

At the first Prague Symposium P. Alexandroff [1] asked the following question: *Which spaces can be represented as images (or inverse images) of nice spaces by nice maps*? Many interesting works have been done to answer this question and Arhangelskii [5] gave a good survey on the results obtained by 1966 and posed many interesting problems, some of which were answered since then. As for generalized metric spaces efforts along this idea coincide with those to characterize various generalized metric spaces in a unified manner, i.e., as the images (or inverse images) of metric spaces by adequate maps. Typical results in this respect are Theorems 2 and 3. Besides we can characterize many spaces as images of metric spaces as partially seen in the following theorem. (The left hand of equality shows the characterized space, and the right hand the used map. * denotes that the map is multi-valued, and † that the space is characterized as the image of 0-dimensional metric space and "0-dimensional" is essential. We need regularity for the characterizations of M_2 and 1st countable M_2 The reader is referred to the papers in parentheses for the definitions of used maps.)

Theorem 10. *Semi-stratifiable = semi-stratifiable* (Nagata [55]), *semi-metric = = a certain condition* (Heath [21]), *σ = σ and one-to-one* (Nagami [76]), *σ = σ-locally finite* (Michael [39]), *σ = s. perfect** (Nagata [52]), *stratifiable = stratifiable* (Nagata [55]), *Nagata = a certain condition* (Heath [21]), *M_2 = q-open and q-closed*†* (Nagata [55]), *M_2 and first countable = almost open and q-closed†* (Nagata [55]), *developable = open π* (Arhangelskii [4], Heath [21]), *M^* = perfect and Y-countably compact** (Nagata [52]), *strict p = open π and bi-Y-compact** (Nagata [55]).

Dealing with the same problem from a little different point of view, many efforts also have been done to characterize the images of metric spaces by given (popular) maps though the image spaces are not necessarily nice as generalizations of metric spaces. As well-known, the images of metric spaces by open (S. Ponomarev [59], S. Hanai [71]), pseudoopen (Arhangelskii [3]), quotient (Franklin [19]), bi-quotient (Michael [39]), closed (N. Lašnev [26]), open compact (Arhangelskii [5]) and open s-maps (Ponomarev [59]) are characterized. In this respect T. Hoshina [28] recently answered Arhangelskii's problem [5] by *characterizing the images of metric spaces by quotient s-maps.* There are also works to *characterize images of M-spaces,* e.g. *by open* (Nagata [49]), *quotient and bi-quotient* (Nagata [47]), *pseudoopen* (T. Rishel [60]) *and almost open* (T. Chiba [14]) *maps* as well as *the images of paracompact M-spaces by open maps* (H. Wicke [67]). (According to recent news Morita and Rishel have characterized images of M-spaces by closed maps.) The perfect images of M-spaces were recently characterized by Nagata [55] as follows.

Theorem 11. *X is an M^*-space iff it is the image of an M-space by a perfect map.*

Theorem 11 was proved using the characterization of an M^*-space in Theorem 10. Unfortunately, the same method will not be applicable for characterization of perfect images of p-spaces. In this connection J. M. Worrell [77] answered Arhangelskii's question [5] in the negative by giving an example of a p-space whose perfect image is not p. It was also announced by Worrell and Wicke that the perfect image of a θ-refinable p-space is p. We can characterize σ-spaces by developing the characterization in Theorem 10 as follows:

Definition 3. Let (X, X') be a pair of a space X and its subspace X'. If X has a sequence $\{\mathscr{U}_i\}$ of locally finite open covers satisfying

(H) for every $x \in X'$ and every nbd V of x in X, there is $U \in \bigcup_{i=1}^{\infty} \mathscr{U}_i$ such that $x \in U \subset V$, then (X, X') is called a *partially metric space* (*or half-metric space*).

Theorem 12 (Nagata [55]). *A regular space Y is a σ-space iff there is a half-metric space (X, X') and a perfect map f from X onto Y such that $f(X') = Y$ iff there is a half-metric space (X, X') and a closed map f from X onto Y such that $f(X') = Y$.*

5. Metrization and other aspects

As for metrization of generalized metric spaces, a basic rule is that metrizability of a space is equal to a condition of type 1 plus a condition of type 2 (plus some additional condition). This rule was first observed by Borges [9], [10] and Okuyama [56]; the latter proved that metrizability $= M + \sigma +$ paracompact. Their theorems were improved by several mathematicians, especially by F. Slaughter, who observed that metrizability $= M + \sigma + T_2$. Probably the best result along this line of efforts is the following theorem due to T. Shiraki [61]. (See also [73].)

Definition 4 ([31]). X is a *wM-space* if it has a sequence $\{\mathcal{U}_i\}$ of open covers satisfying: If $x_i \in S^2(x, \mathcal{U}_i)$, $i = 1, 2, \ldots$, then $\{x_i\}$ clusters.

Definition 5 ([62]). A collection $\mathcal{U}$ of closed sets is a *ct-net* if for every $x \in X$, $\bigcap\{U \mid x \in U \in \mathcal{U}\} = \{x\}$. A space with a σ-closure preserving *ct*-net is $\sigma^{\#}$.

Every M^*-space is wM, every wM-space is $w\Delta$, and every semi-stratifiable space is $\sigma^{\#}$, which coincides with Hodel's *α-space*.

Theorem 13. *A T_2-space X is metrizable iff it is wM and $\sigma^{\#}$.*

Another important factor of metrizability is *point-countable base*. V. Filippov proved in his celebrated paper [18] that *every paracompact M-space with a point-countable base is metrizable*. This theorem was generalized by Nagata [48], Slaughter [63], Shiraki [61], J. Suzuki and others. Let us give a version of their theorem.

Definition 6. A collection $\mathcal{U}$ of open sets is called a *p-base* if for each $x \in X$, $\bigcap\{U \mid x \in U \in \mathcal{U}\} = \{x\}$.

This concept is given various names, e.g. pseudo-base, T_1-cover, separating open cover, etc. Since it is proving useful as a generalization of a base, a standard terminology should be decided.

Theorem 14. *A T_2-space X is metrizable iff it is M^* and has a point-countable p-base.*

The extension of Filippov's theorem in another direction was done by Michael [39]. The following is T. Shiraki's [61] improved version. (Essentially the same theorem is given also in [78].)

Definition 7. X is a *Σ-space* if it has a sequence $\{\mathscr{U}_i\}$ of locally finite closed covers satisfying: If $x_i \in C(x, \mathscr{U}_i) = \bigcap\{U \mid x \in U \in \mathscr{U}_i\}$, $i = 1, 2, \ldots$, then $\{x_i\}$ clusters.

This notion due to K. Nagami [44] is interesting as it generalizes both M^* and regular σ-spaces.

Theorem 15. *X is metrizable iff it is collectionwise normal Σ and has a point-countable base.*

This theorem is easily derived from the following theorem of Michael and Slaughter [78], which is interesting in its own right.

Theorem 16. *Every Σ-space X with a point-countable p-base is σ.*

Each of the above three metrization theorems implies none of the others; a more unified theory is desirable in this aspect.

"Developable space" is a classical example of generalized metric space. Some authors chose this space (instead of metric space) as their starting point to study generalized metric spaces in their papers. In this connection generalizations of paracompactness like *σ-paracompactness* ([5]) and *θ-refinability* ([69]) are proving useful, because general developable spaces satisfy only these conditions weaker than paracompactness.

Ishiwata [33], for example, has done an extensive investigation on inverse images of developable spaces by perfect and quasi-perfect maps. The following theorem may be compared with Theorem 13.

Theorem 17 (Burke [12]). *A regular space is developable iff it is $w\Delta$ and σ^*.*

Čoban [75] and Hodel [27] also got interesting results about developability of spaces.

The importance of a generalized metric space largely depends on how beautifully theorems can be extended from metric spaces. Many of the previously mentioned theorems imply such extensions. Borges [8] extended the classical Dugundji's extension theorem on map of a metric space to a stratifiable space. Lašnev's theorem [35] (given a closed map f from a metric space X into Y, then $Y = \bigcup_{n=0}^{\infty} Y_n$, where each Y_n, $n > 0$ is discrete and $f^{-1}(y)$ is compact for each $y \in Y_0$) was extended by Filippov [18] to paracompact M-space, by R. S. Stoltenberg [64] to normal semi-stratifiable space and by Nagata [53] to a class of spaces including wM-spaces and semi-metric spaces. Nagami [44] generalized Morita's theorem on metric space as follows: *Let X be a paracompact Σ-space and Y a paracompact P-space in the sense of Morita* [40]; *then $X \times Y$ is paracompact.* Extension is not always an easy task. The author's conjecture [45] at the 2nd Prague Symposium: $\dim X \times Y \leqq \dim X +$ $+ \dim Y$ *for paracompact M-spaces*? still remains open while the same is a well-known theorem for metric spaces and compact spaces. There is another interesting aspect. Morita [43] developed a theory on paracompactification of M-space. He defined $\mu(X)$ for every Tychonoff space X as the completion of X relative to its finest uniformity to show that *$\mu(X)$ is paracompact M if X is M* and μX has especially interesting properties.

There are many other generalizations (related to type 1 or 2) of metric spaces which have not been discussed here, M' ([74]), $M^{\#}$ ([62]), $\Sigma^{\#}$ ([39]), *strong* Σ ([44]), Σ^* ([58]), *k-semi-stratifiable* ([37], *submetrizable* (= *contractible onto a metric space* ([34])) to name just a few. The definitions of all generalizations mentioned by now contain some countability in the sense that they are defined in terms of existence of countably many covers, countably many collections of sets, etc. H. Tamano's [65] *elastic space* and J. Vaughan's [66] *linearly stratifiable space* are probably the first attempts to generalize stratifiable space by dropping countability from the definition. All of the discussed spaces were generalizations of *general* metric space. There are interesting generalizations of special metric spaces, e.g. Worell and Wicke [69] generalized complete metric space, and Michael [38] generalized separable metric space. However, for those spaces and others another survey should be given since we have already exceeded the expected length of lecture.

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