E. H. Kronheimer
Very unlatticelike ordered spaces


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All partial orderings \(<\) ("strictly precedes") are to satisfy \(p < q < p \Rightarrow p = q\); the converse implication may, but need not, hold — indeed a point which strictly precedes itself will be called singular. The reflexive relation \(\leq\) ("precedes") is associated with \(<\) in the usual way (\(p \leq q \iff p < q \text{ or } p = q\)); and we write

\[
L(q) = \{x \mid x < q\}, \quad L[Q] = \bigcup_{q \in Q} L(q).
\]

A non-void subset \(D\) of a partially ordered set is directed (resp. strictly directed) if, given two points in \(D\), there exists a point in \(D\) succeeding (resp. strictly succeeding) both; \(D\) is a strict ideal if it is strictly directed and contains all the predecessors of each of its points. Call \(L[Q]\) the set generated by \(Q\); then every strictly directed set generates a strict ideal, and every strict ideal generates itself. Any set of the form \(L(q)\) is called a principal strict ideal.

Borrowing a term from Michael [2], we call a partially ordered set a cushion if it satisfies any of the following equivalent conditions:

(a) Every point has a strict predecessor; and, whenever \(p_1, p_2 < r\), there exists \(q\) satisfying \(p_1, p_2 < q < r\).

(b) Every directed subset generates a strict ideal.

(c) Every principal strict ideal is a strict ideal.

We equip each cushion with the topology determined by the base \(\{\langle p, q \rangle \mid p < q\}\), where \(\langle p, q \rangle = \{x \mid p < x \leq q\}\). The singular points of a cushion are then its isolated points, and it is Hausdorff if and only if \(p = q\) whenever \(L(p) = L(q)\). A subset \(S\) of the cushion \(X\) is called a subcushion of \(X\) if \(S\) (with the restricted ordering) is itself a cushion whose topology coincides with its topology as a subspace of \(X\). A function \(f\) between cushions is called a cushion map if it is continuous and \(p < q\) implies \(f(p) < f(q)\).

Example 1. Let \(E\) be a normal \(T_1\)-space. (Somewhat weaker separation axioms are in fact sufficient.) Let \(cE\) denote the collection of its open subsets other than \(\emptyset\) and \(E\), ordered by putting \(U < V\) iff \(U^- \subset V\). Then \(cE\) is a Hausdorff cushion which is non-singular if and only if \(E\) is connected. If \(E, F\) are normal \(T_1\)-spaces
and \( \theta : E \to F \) is a closed continuous surjection, then \( c\theta : cF \to cE \), where \( c\theta(W) = \theta^{-1}[W] \), is a cushion map.

Example 2. Let \( \mathbb{R}^- \) denote the non-positive real numbers and \( M \) be a pseudometric space. Let \( kM \) denote the set \( M \times \mathbb{R}^- \), ordered by putting \((m_1, r_1) < (m_2, r_2)\) iff \( d(m_1, m_2) < r_2 - r_1 \). Then \( kM \) is a non-singular cushion which is Hausdorff if and only if \( M \) is Hausdorff (i.e., metric). If \( M, N \) are pseudometric spaces and the function \( \varphi : M \to N \) satisfies \( d(\varphi(m_1), \varphi(m_2)) < \lambda d(m_1, m_2) \) for some fixed positive number \( \lambda \), then the function \( k\varphi : kM \to kN \), where \( k\varphi(m, r) = (\varphi(m), \lambda r) \), is a cushion map.

A cushion in which every strict ideal is principal is called \textit{complete}. Complete cushions have some pleasant properties. Let us, for instance, say that a net \((s_d)_{d \in D}\) (on the directed set \( D \)) in a partially ordered space is \textit{increasing} if \( d \leq e \) implies \( s_d \leq s_e \). Then a cushion \( X \) is complete (resp. Hausdorff) if and only if every increasing net in \( X \) has at least (resp. at most) one limit point. Again: every closed subcushion of a complete cushion is complete, and every complete subcushion of a Hausdorff cushion is closed. The results we shall establish here are two more specific ones.

**Theorem 1.** The cushion \( cE \) is complete if and only if the topological space \( E \) is compact.

**Proof.** Assume first that \( E \) is compact, and let \( \mathcal{P} \) be an ideal in \( cE \); then \( P^* = \bigcup_{P \in \mathcal{P}} P \) is open and non-void. Suppose \( Q \) is a non-void open set such that \( Q^- \subseteq P^* \). Since \( \mathcal{P} \) covers \( Q^- \), so does some finite subcollection \( \{P_1, \ldots, P_m\} \) of \( \mathcal{P} \). Since \( \mathcal{P} \) is directed, some member of \( \mathcal{P} \) contains all the \( P_i \) and hence contains \( Q^- \). It follows that \( Q \in \mathcal{P} \). Since \( E \notin \mathcal{P} \), this argument (with \( Q = E \)) shows that \( P^* \neq E \); so \( P^* \in cE \). It also shows that \( L(P^*) \subseteq \mathcal{P} \). On the other hand, if \( P \in \mathcal{P} \), then \( P < P' \in \mathcal{P} \) for some \( P' \), so that \( P^- \subseteq P' \subseteq P^* \): therefore \( P \in L(P^*) \). It follows that \( \mathcal{P} = L(P^*) \), and hence that \( cE \) is complete.

To prove the converse, assume \( E \) has an open cover \( \mathcal{U} \) with no finite refinement. Let \( \mathcal{V} \) denote the collection of all non-void sets expressible as finite unions of members of \( L[\mathcal{U}] \). Then \( \mathcal{V} \) is a directed subset of \( cE \) which fails to generate a principal strict ideal; for if \( L[\mathcal{V}] = L(W) \), where \( W \in cE \), then (since \( \mathcal{V} \) is actually strictly directed) each member of \( \mathcal{V} \) is a subset of \( W \), contradicting the fact that \( \mathcal{V} \) covers \( E \).

**Theorem 2.** The cushion \( kM \) is complete if and only if the pseudometric space \( M \) is complete.

**Proof.** Suppose \( M \) is a complete pseudometric space and \( P \) is a strict ideal in \( kM \). Let \( r^* = \sup \{r \mid (x, r) \in P \text{ for some } x \in M \} \),
and choose \( x_0, x_1, \ldots \) in \( M \) such that \( (x_n, r^* - 2^{-n}) \in P \) for each \( n \). Then \( (x_n) \) is a Cauchy sequence; and \( P = L(x^*, r^*) \), where \( x^* \) is a limit of \( (x_n) \).

Conversely, suppose that \( kM \) is a complete cushion and \( (y_n) \) is a Cauchy sequence. Define

\[
s_n = -2 \sup_{k \geq 0} d(y_n, y_{n+k}).
\]

The set \( \{(y_0, s_0), (y_1, s_1), \ldots\} \) is directed and therefore generates a strict ideal: call this \( L(q) \). Then \( q \) must be of the form \((y, 0)\), and \( y \) must be a limit of \( (y_n) \).

If \( X \) is a complete Hausdorff cushion and \( f : X \to X \) a cushion map satisfying \( a < f(a) \) for some \( a \), then \( f \) has a fixed point. (To construct it, put \( a_0 = a, a_{n+1} = f(a_n) \). If the directed set \( \{a_0, a_1, \ldots\} \) generates \( L(b) \), then \( f(b) = b \).) Theorem 2 shows that this result includes the Banach fixed-point theorem: for if \( \phi : M \to M \) satisfies \( d(\phi(m_1), \phi(m_2)) < \lambda d(m_1, m_2) \), where \( \lambda < 1 \), and if \( m \) is any point of \( M \), then \( (m, r) \) strictly precedes \( k_\lambda \phi(m, r) \) in the cushion \( kM \) for all sufficiently large \(-r\); and if \( k_\lambda \phi \) has a fixed point, so has \( \phi \). (It may be noted that, working with reflexive orderings, one obtains, instead of propositions about the (complete) cushion \( kM \), closely analogous propositions about (Dedekind \( \sigma \)-complete) ordered sets. See [1].)

References


BIRKBECK COLLEGE, UNIVERSITY OF LONDON