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HARMONIC ANALYSIS AND TOPOLOGY

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The mathematical discipline that goes by the name of abstract harmonic analysis has close points of contact with set-theoretic topology, in a variety of directions.

First let us say briefly what harmonic analysis is. The classical theories of Fourier series and Fourier integrals are a highly developed branch of analysis. They have attracted the best mathematical minds for three centuries and may be regarded as part of the tradition of the human race. Given an integrable function f on $[-\pi, \pi[$, we form its Fourier coefficients

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-ikt)\, dt, \quad k = 0, \pm 1, \pm 2, \ldots,$$

and we may ask if the Fourier coefficients determine the function, how to reconstruct f from its Fourier coefficients, what relations exist between the size of f and the size of its Fourier coefficients. Or we can look at closed subspaces of $\mathfrak{L}_p([-\pi, \pi[)$ $(1 \leqq p \leqq \infty)$ invariant under translation modulo 2π and attempt to classify all of these subspaces, perhaps in terms of the Fourier coefficients. Similarly for $f \in \mathfrak{L}_1(-\infty, \infty)$, we can form the Fourier transform

$$\hat{f}(y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \exp(-ixy)\, dx$$

and ask precisely the same questions.

In abstract harmonic analysis, one replaces the group $\mathbf{T} = [-\pi, \pi[$ (a compact Abelian group under addition modulo 2π and the usual topology) and $\mathbf{R}$ (a locally compact, noncompact, Abelian group under addition and the usual topology) by a locally compact group G. In studying the structure of such groups, it is plain that the topology must play a decisive role. Thus even while one takes the first step into harmonic analysis, one inescapably draws upon topology.

A locally compact group G admits an (essentially) unique Haar measure, which at first is constructed as a translation-invariant nonnegative linear functional L on the space $\mathfrak{C}_{00}(G)$ of all continuous complex-valued functions on G having compact supports. The shortest proof of the existence of L uses Tihonov's theorem, although it can be avoided by an ingenious construction due to Henri Cartan. Using L to construct a Borel measure λ on G for which $L(f) = \int_G f(t)\,\mathrm{d}\lambda(t)$ for all $f \in \mathfrak{C}_{00}(G)$, we can construct function spaces $\mathfrak{L}_p(G, \lambda)$ for $0 < p \leqq \infty$, and proceed to study these function spaces. We can also consider the space $\mathfrak{C}_0(G)$ consisting of all continuous complex-valued functions on G that are arbitrarily small in absolute value outside of compact sets (the completion of $\mathfrak{C}_{00}(G)$ in the uniform metric) and construct its conjugate space $\mathbf{M}(G)$, the space of all complex-valued, countably additive, regular, Borel measures on G. Abstract harmonic analysis is primarily the study of the structure of the spaces $\mathfrak{L}_p(G)$ and of $\mathbf{M}(G)$ and of entities which arise in studying them.

For the sake of staying within reasonable boundaries, we will suppose henceforward that our groups G are either compact or Abelian and locally compact. For locally compact Abelian groups, the functions $t \mapsto \exp(ikt)$ on $\mathbf{T}$ and $x \mapsto$ $\mapsto \exp(ixy)$ on $\mathbf{R}$ are analogized by the continuous characters of G, which are the continuous complex functions χ of absolute value 1 on G such that $\chi(xy) = \chi(x)\,\chi(y)$ for all $x, y \in G$. The set $\mathbf{X}$ of all continuous characters is an Abelian group under pointwise multiplication, and topologized by the compact-open topology, it is also locally compact. The celebrated duality theorem of Pontryagin and van Kampen asserts that the character group of $\mathbf{X}$ is G; every continuous character of $\mathbf{X}$ has the form $\chi \mapsto \chi(a)$ for some fixed $a \in G$, and the topology of G as characters of $\mathbf{X}$ is its original topology. The *Fourier transform* of $f \in \mathfrak{L}_1(G)$ is the function $\hat{f}$ on $\mathbf{X}$ such that

$$\hat{f}(\chi) = \int_G f(t)\,\overline{\chi(t)}\,\mathrm{d}\lambda(t) \quad \text{for all} \quad \chi \in \mathbf{X}\,.$$

Given a measure $\mu \in \mathbf{M}(G)$, its *Fourier-Stieltjes transform* is the function $\hat{\mu}$ on $\mathbf{X}$ such that

$$\hat{\mu}(\chi) = \int_G \overline{\chi(t)}\,\mathrm{d}\mu(t) \quad \text{for all} \quad \chi \in \mathbf{X}\,.$$

For compact non-Abelian groups G, the role of the character group is taken over by irreducible unitary representations of G. Given a complex Hilbert space H, we denote by $\mathfrak{B}(H)$ the algebra of all bounded linear operators on H and by $\mathfrak{U}(H)$ the group of unitary operators on H. Now consider a homomorphism $U\colon x \mapsto U_x$ of G into $\mathfrak{U}(H)$ such that $U_e = I$. If there are no proper closed subspaces J of H such that $U_x(J) \subset J$ for all $x \in G$, U is called *irreducible.* If the map $x \mapsto \langle U_x\xi, \eta\rangle$ is continuous for all $\xi, \eta \in H$, then U is called *continuous.* All continuous unitary irreducible representations of a compact group G are finite-dimensional. Given two representations U and U' of G, with representation spaces H and H', we say that U and U' are *equivalent* if there is a linear isometry A of H onto H' such that $AU_xA^{-1} =$ $= U'_x$ for all $x \in G$. Let Σ denote the set of all equivalence classes of continuous unitary irreducible representations of G. That is, an element σ of Σ consists of a representation $U^{(\sigma)}$ and all representations equivalent to $U^{(\sigma)}$. We call Σ the *dual* *object of G.* To define Fourier transforms we need also a *conjugation D* on our Hilbert space H; i.e., an additive map D of H onto itself such that $D(\alpha\xi) = \bar{\alpha}\, D(\xi)$ for all $\alpha \in \mathbf{C}$ and $\xi \in H$. (Such conjugations all have the form $D(\Sigma\alpha_\iota\zeta_\iota) = \Sigma\bar{\alpha}_\iota\zeta_\iota$ for some orthonormal basis in H.) Now for every $\sigma \in \Sigma$, choose some $U^{(\sigma)} \in \sigma$ with representation space H_σ and a conjugation D_σ on H_σ. Let d_σ be dim (H_σ). For $f \in \mathfrak{L}_1(G)$, we define the *Fourier transform* $\hat{f}$ as the $\mathfrak{B}(H_\sigma)$-valued function on Σ such that

$$\langle \hat{f}(\sigma)\,\xi, \eta\rangle = \int_G \langle D_\sigma U_x^{(\sigma)} D_\sigma \xi, \eta\rangle\, f(x)\, d\lambda(x)$$

for all $\sigma \in \Sigma$ and $\xi, \eta \in H_\sigma$. For $\mu \in \mathbf{M}(G)$, the *Fourier-Stieltjes* transform $\hat{\mu}$ is defined similarly:

$$\langle \hat{\mu}(\sigma)\,\xi, \eta\rangle = \int_G \langle D_\sigma U_x^{(\sigma)} D_\sigma \xi, \eta\rangle\, d\mu(x)\,.$$

Plainly $\hat{f}(\sigma)$ and $\hat{\mu}(\sigma)$ depend upon the particular choice of $U^{(\sigma)}$ and D_σ, but all such choices yield unitarily equivalent operators, and for all known norms of transforms and all presently studied structural properties of $\mathfrak{L}_p(G)$ and $\mathbf{M}(G)$, this is sufficient.

We also introduce some norms into the space of Fourier transforms. Given a finite-dimensional Hilbert space H of dimension d, and $A \in \mathfrak{B}(H)$, write the positive-definite operator AA^* as $\sum_{j=1}^{d} a_j P_j$, where the P_j's are 1-dimensional projections. For $1 \leqq p < \infty$, define the *von Neumann norm* $\|A\|_{\Phi_p}$ as $(\sum_{j=1}^{d} a_j^{p/2})^{1/p}$ and $\|A\|_{\Phi_\infty}$ as $\max_{1 \leqq j \leqq d} a_j^{1/2}$. The norm $\|A\|_{\Phi_\infty}$ is the operator norm, and $\|A\|_{\Phi_2}$ is Frobenius's trace norm $[\mathrm{tr}\,(A^*A)]^{1/2}$. Define

$$\|\hat{f}\|_p = (\sum_{\sigma\in\Sigma} d_\sigma \|\hat{f}(\sigma)\|_{\Phi_p}^p)^{1/p} \quad (1 \leqq p < \infty)$$

and

$$\|\hat{f}\|_\infty = \sup_{\sigma\in\Sigma} \|\hat{f}(\sigma)\|_{\Phi_\infty}\,.$$

The Weyl-Peter theorem asserts that if $f \in \mathfrak{L}_2(G)$, then $\|f\|_2 = \|\hat{f}\|_2$, and the Riemann-Lebesgue lemma that $\|\hat{f}(\sigma)\|_{\Phi_\infty}$ is arbitrarily small off of a finite set in Σ.

The function spaces $\mathfrak{L}_p(G)$ $(0 < p \leqq \infty)$ admit isometries defined by translation. For $a \in G$ and any function f on G, define ${}_af$ and f_a by ${}_af(x) = f(ax)$ and $f_a(x) = f(xa)$. A space $\mathfrak{S}$ of functions is called (left, right) invariant if $f \in \mathfrak{S}$ implies $({}_af, f_a) \in \mathfrak{S}$ for all $a \in G$. A main problem of harmonic analysis is the classification of all closed invariant subspaces of the spaces $\mathfrak{L}_p(G)$. For compact groups G and $1 \leqq p \leqq \infty$, this classification is complete. For $1 \leqq p < \infty$, every two-sided invariant subspace of $\mathfrak{L}_p(G)$ is defined by the vanishing of $\hat{f}(\sigma)$ for all σ in a certain subset of Σ. Slight additional complications arise for $p = \infty$ and for one-sided invariant subspaces.

For noncompact Abelian groups G, the invariant subspaces are completely known only for $\mathfrak{L}_2(G)$. For $0 < p < 1$, nothing is known for any infinite group G.

The space of measures $\mathbf{M}(G)$ is a Banach algebra under *convolution:*

$$\mu * \nu(E) = \int_G \mu(Et^{-1})\, \mathrm{d}\nu(t) = \int_G \nu(t^{-1}E)\, \mathrm{d}\mu(t)$$

for all Borel subsets E of G. Of course $\mathfrak{L}_1(G)$ is also a Banach algebra under convolution:

$$f * g(x) = \int_G f(xy)\, g(y^{-1})\, \mathrm{d}y\,.$$

Closed ideals in $\mathfrak{L}_1(G)$ are exactly the closed invariant subspaces of $\mathfrak{L}_1(G)$ (left, right, and two-sided, respectively), so the ideal structure of $\mathfrak{L}_1(G)$ is coextensive with its invariant subspace structure.

The algebra $\mathbf{M}(G)$ is very little understood, in spite of considerable recent progress. For locally compact Abelian G, one has the Gel'fand theory of its maximal ideals, and its compact structure space $\mathscr{S}$. However, the topology of $\mathscr{S}$ is formidably complicated, as has been long known, and there seems no hope of unravelling its puzzles completely in the near future. See for example Yu. A. Šreĭder [15] and Hewitt and Kakutani [5], [6]. Further progress in understanding $\mathbf{M}(G)$ may depend upon brilliant applications of functional analysis like those made by J. L. Taylor [16], [17], [18], and N. Th. Varopoulos [19], [20], [21], (the latter to be sure in part for other purposes). In the author's opinion, however, decisive progress can be expected only through refined study of the arithmetic and analytic properties of certain perfect subsets of G.

In a brief survey it is plainly impossible to give a complete picture of the role of topology in harmonic analysis. We will therefore limit ourselves to two examples. The first is the theory of *lacunary sets,* where the theory of topological linear spaces is applied. The basic idea of a lacunary set is this. If f is a function whose Fourier transform vanishes except on a "thin" or "lacunary" set (of characters in the Abelian case, or of representations in the non-Abelian case), then f can be expected to have some extraordinary properties. Ancestors of many such theorems are two theorems of S. Sidon [13], [14]. A *Hadamard set* $\{n_k\}_{k=1}^{\infty}$ of positive integers is defined by the property that for some constant $q > 1$, we have

$$n_{k+1}/n_k \geqq q \quad \text{for all} \quad k\,.$$

Sidon proved that if a real function f is in $\mathfrak{L}_\infty([-\pi, \pi[)$ (or merely in $\mathfrak{L}_1([-\pi, \pi[)$) and if $\hat{f}(n) = 0$ except for $n = \pm n_k$, where the n_k's are in a Hadamard set, then $\sum_{n\in\mathbf{Z}} |\hat{f}(n)| < \infty$ (or merely $\sum_{n\in\mathbf{Z}} |\hat{f}(n)|^2 < \infty$).

We make these theorems the bases of definitions.

Definition. Let G be any compact group, with dual object $\boldsymbol{\Sigma}$. A subset $\mathbf{P}$ of $\boldsymbol{\Sigma}$ is called a *Sidon set* if for every function $f \in \mathfrak{C}(G)$ such that $\hat{f}(\sigma) = 0$ for all σ in in $\boldsymbol{\Sigma} \setminus \mathbf{P}$, we have $\|\hat{f}\|_1 < \infty$.

Definition. For $1 < p < \infty$, a subset $\mathbf{P}$ of $\boldsymbol{\Sigma}$ is called a Λ_p *set* if for every function $f \in \mathfrak{L}_1(G)$ such that $\hat{f}(\sigma) = 0$ for all $\sigma \in \boldsymbol{\Sigma} \setminus \mathbf{P}$, we have $f \in \mathfrak{L}_p(G)$.

Both of these definitions are abstractions from Sidon's theorems. Now we can list a large number of properties equivalent to each of the above properties.

Theorem. *Let* $\mathbf{P}$ *be a subset of* $\boldsymbol{\Sigma}$. *The following assertions are equivalent*:

(i) $\mathbf{P}$ *is a Sidon set*;

(ii) *for every operator function* $(E_\sigma)_{\sigma \in \mathbf{P}}$ *with* $E_\sigma \in \mathfrak{B}(H_\sigma)$ *such that* $\sup_{\sigma \in \mathbf{P}} \|E_\sigma\|_{\Phi_\infty} < < \infty$, *there is a measure* $\mu \in \mathbf{M}(G)$ *such that* $\hat{\mu}(\sigma) = E_\sigma$ *for all* $\sigma \in \mathbf{P}$;

(iii) *for every* $(E_\sigma)_{\sigma \in \mathbf{P}}$ *as in* (ii) *such that* $\|E_\sigma\|_{\Phi_\infty}$ *is arbitrarily small outside of finite sets, there is a function* $f \in \mathfrak{L}_1(G)$ *such that* $\hat{f}(\sigma) = E_\sigma$ *for all* $\sigma \in \boldsymbol{\Sigma}$;

(iv) *if* $f \in \mathfrak{L}_\infty(G)$ *and* $\hat{f}(\sigma) = 0$ *for all* $\sigma \in \boldsymbol{\Sigma} \setminus \mathbf{P}$, *then* $\|\hat{f}\|_1 < \infty$;

(v) *there is a constant* $\varkappa$ *such that* $\|\hat{f}\|_1 \leqq \varkappa \|f\|_\infty$ *for all* $f \in \mathfrak{L}_\infty(G)$ *such that* $\hat{f}(\sigma) = 0$ *for* $\sigma \in \boldsymbol{\Sigma} \setminus \mathbf{P}$;

(vi) *the same as* (v) *for continuous* f;

(vii) *the same as* (v) *for* f *such that* $\hat{f}(\sigma) \neq 0$ *only for a finite set of* $\sigma \in \boldsymbol{\Sigma}$;

(viii) *for every unitary operator-valued function* $U_\sigma \in \mathfrak{U}(H_\sigma)$ *defined for* $\sigma \in \mathbf{P}$, *there is a* $\mu \in \mathbf{M}(G)$ *such that* $\sup_{\sigma \in \mathbf{P}} \|U_\sigma - \hat{\mu}(\sigma)\|_{\Phi_\infty} < 1$.

Obviously the proof of such a string of equivalences is long. However at bottom it is merely an exercise in functional analysis. The open mapping theorem and the Hahn-Banach theorem are all that one really needs. Such applications of functional analysis occur repeatedly in harmonic analysis.

A similar theorem applies to Λ_p sets: they may be characterized in a variety of ways using the techniques of functional analysis. It can be shown that every Sidon set is a Λ_p set for all $p > 1$. For infinite Abelian G, the character group always contains a set $\mathbf{P}$ that is a Λ_p set for all $p > 1$ but is not a Sidon set, as was recently proved by R. E. Edwards, K. A. Ross, and the author [3].

Our second example applies the Čech-Stone compactification, or if you like, the theory of ultrafilters, to a problem in harmonic analysis. Let us consider a compact non-Abelian group G. Closed ideals in the algebra $\mathfrak{L}_1(G)$, as noted above, are completely known. In particular, the maximal closed two-sided ideals in $\mathfrak{L}_1(G)$ all have the form $\mathfrak{J}_\tau = \{f \in \mathfrak{L}_1(G) : \hat{f}(\tau) = 0\}$ for some fixed $\tau \in \boldsymbol{\Sigma}$. The simple quotient algebra $\mathfrak{L}_1(G)/\mathfrak{J}_\tau$ is easily seen to be isomorphic to $\mathfrak{B}(H_\tau)$ and so is finite-dimensional. Now consider the measure algebra $\mathbf{M}(G)$. Does the same phenomenon persist in this large algebra? That is, if $\mathbf{I}$ is a maximal two-sided ideal in $\mathbf{M}(G)$, is the simple algebra $\mathbf{M}(G)/\mathbf{I}$ finite-dimensional? Of course this question is trivial if G is Abelian. For Abelian G, $\mathbf{M}(G)$ is commutative, and the Gel'fand-Mazur theorem shows that all algebras $\mathbf{M}(G)/\mathbf{I}$ are isomorphic with $\mathbf{C}$. Negative answers for some G are obtained, however, with the use of ultrafilters. The construction is joint work of Dusa McDuff and the author [7].

Let us sketch the construction. Consider an infinite index set I and a countably infinite dissection $\mathscr{D} = \{A_k\}_{k=1}^{\infty}$ of I into subsets, some possibly void. For each $\iota \in A_k$ consider a Hilbert space H_ι of dimension k, and form the product algebra $\mathbf{P}_{\iota \in I} \mathfrak{B}(H_\iota)$. In this algebra, single out the elements $E = (E_\iota)_{\iota \in I}$ such that $\|E\| = \sup_{\iota \in I} \|E_\iota\|_{\Phi_\infty} < \infty$, and call this Banach algebra $\mathfrak{E}(I, \mathscr{D})$. The algebra $\mathfrak{E}(I, \mathscr{D})$ is a noncommutative analogue of the algebra $\mathfrak{C}_b(I)$ consisting of all bounded complex-valued functions on I, to which of course it reduces if $I = A_1$. All of the maximal two-sided ideals in $\mathfrak{E}(I, \mathscr{D})$ can be constructed by using the Čech-Stone compactification βI of I, where we regard I as a discrete space. To do this, we associate a certain real-valued function with every $E = (E_\iota)_{\iota \in I}$ in $\mathfrak{E}(I, \mathscr{D})$. Define $\psi_E(\iota)$ as $k^{-1/2}\|E_\iota\|_{\Phi_2}$ for all $\iota \in A_k$. It is trivial that $0 \leqq \psi_E(\iota) \leqq \|E_\iota\|_{\Phi_\infty} \leqq \|E\|_\infty < \infty$, so that ψ_E is a bounded real-valued function on I. By the defining property of βI, ψ_E admits a continuous extension over βI, which we will continue to call ψ_E. For every point $x \in \beta I$, we define

$$\mathfrak{J}_x = \{E \in \mathfrak{E}(I, \mathscr{D}) : \psi_E(x) = 0\} .$$

Then $\mathfrak{J}_x$ is a maximal two-sided ideal in $\mathfrak{E}(I, \mathscr{D})$, and every maximal two-sided ideal in $\mathfrak{E}(I, \mathscr{D})$ is $\mathfrak{J}_x$ for some $x \in \beta I$. This is a special case of a theorem of F. B. Wright [22]. The theorem of Pospíšil [10] shows that there are $\exp(\exp(\operatorname{card}(I)))$ such ideals: plainly if $x \neq x'$, then $\mathfrak{J}_x \neq \mathfrak{J}_{x'}$.

As is well known from the work of P. S. Aleksandrov and others, the points of βI are in one-to-one correspondence with the ultrafilters on I. In fact, we may regard βI as the set of all ultrafilters on I. Let x stand for an ultrafilter on I. If some set A_k belongs to x, then the quotient algebra $\mathfrak{E}(I, \mathscr{D})/\mathfrak{J}_x$ is isomorphic with $\mathfrak{B}(H)$ where H is a k-dimensional Hilbert space. On the other hand, if x contains no set A_k, then $\mathfrak{E}(I, \mathscr{D})/\mathfrak{J}_x$ is infinite-dimensional and is in fact nonseparable. Thus the algebras $\mathfrak{E}(I, \mathscr{D})$ admit infinite-dimensional simple homomorphs provided that the dimensions of the spaces H_ι are unbounded.

Given a metric space Y and a subset S of Y, *we say that* S *is scattered if* $\inf\{\varrho(x, y) : x, y \in S, x \neq y\} > 0$. *Using Frolík's construction of sums of ultrafilters* [4], *we can construct an* x *for which* $\mathfrak{E}(I, \mathscr{D})/\mathfrak{J}_x$ *contains a very large scattered set. The precise result is the following. Let* $\mathfrak{m}$ *be any infinite cardinal, and* I *a set of cardinal number* $\mathfrak{m}$. *Then there is a decomposition* $\mathscr{D} = \{A_k\}_{k=1}^{\infty}$ *of* I *and an ultrafilter* x *in* I *such that the quotient algebra* $\mathfrak{E}(I, \mathscr{D})/\mathfrak{J}_x$ *contains a scattered set of cardinal number* $\exp(\mathfrak{m})$.

To obtain results like this for measure algebras $\mathbf{M}(G)$, we use Sidon sets. Suppose for example that G is a product $\mathbf{P}_{\iota \in I} \mathfrak{U}(H_\iota)$, where each finite-dimensional unitary group is given its usual [compact] topology. For $U = (U_\iota) \in G$, and a fixed $\Theta \in I$, the map $U \mapsto U_\Theta$ is a continuous irreducible unitary representation of G. Let π_Θ be the element of Σ containing this representation. It can be shown that the set $\{\pi_\iota : \iota \in I\}$ is a Sidon set in Σ. Thus the mapping $\mu \mapsto (\hat{\mu}(\iota))_{\iota \in I}$ of $\mathbf{M}(G)$ into $\mathfrak{C}(I, \mathscr{D})$ is an *on*to mapping, and all of the previous paragraph applies to $\mathbf{M}(G)$. The mapping

$$\mathbf{M}(G) \mapsto \mathfrak{C}(I, \mathscr{D}) \mapsto \mathfrak{C}(I, \mathscr{D})/\mathfrak{J}_x$$

is a homomorphism, and for x as above the image is not only infinite-dimensional but contains very large scattered sets. By a closer study of the structure of ultrafilters, it might well be possible to distinguish yet more refinements in simple homomorphs of $\mathbf{M}(G)$.

In conclusion we mention some of the literature on harmonic analysis. The classical treatises of Zygmund [23] and Bari [1] deal exhaustively with the groups $\mathbf{T}$, $\mathbf{Z}$, and $\mathbf{R}$, and in less detail with $\mathbf{T}^m$ and $\mathbf{R}^m$ ($m = 2, 3, \ldots$). They do not have the group-theoretic point of view. R. E. Edwards [2] has written a monograph on Fourier series from the group-theoretic viewpoint, which serves as a fine bridge between the classical and abstract treatment of the subject. A similar attitude is adopted in the book [9] of Katznelson, although his treatment is terser and more technical than Edwards's. A sophisticated and relatively short exposition of analysis on groups has been given by Rudin [12]. Reiter [11] pursues a number of special topics, also mostly on groups. The author and Kenneth A. Ross [8] have written a two-volume monograph on abstract harmonic analysis exclusively from the group-theoretic point of view. A study of either Rudin or Hewitt and Ross should probably be preceded by a close reading of Edwards, if one is not to lose sight of the woods for the multitude of trees.

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