Evert Wattel

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A GENERAL FIXED POINT THEOREM

E. WATTEL

Amsterdam

In general topology there exist some fixed point theorems for contracting mappings. Their common characteristic is that they guarantee uniqueness of the fixed point by means of a principle of contraction relative to the metric of the space. In this paper we shall describe yet another fixed point theorem of the same sort, originating from a problem in differential equations, and we shall give a generalization to uniform spaces. First we shall state two well-known contraction theorems and we shall then prove the main theorem in a metric space and a uniform space separately. Eventually we shall show that Theorem 1 is a special case of Theorem 3, but we shall not include a similar proof for Theorem 2.

Theorem 1. (Banach). *Let (X, ϱ) be a complete metric space and let ϕ be a mapping from X into X such that there exists a positive real number α less than 1 with the property that $\varrho(\phi(x), \phi(y)) \leqq \alpha\, \varrho(x, y)$ for all x and y in X, then X contains one and only one point x_ϕ for which $\phi(x_\phi) = x_\phi$ holds.*

Theorem 2. *Let (X, ϱ) be a metric space and let ϕ be a mapping from X into itself such that $\overline{\phi(X)}$ is compact and $\varrho(\phi(x), \phi(y)) < \varrho(x, y)$ for all $x, y \in X$, then X contains one and only one fixed point relative to the mapping ϕ.*

Convention. If X is a space and ϕ is a mapping from X into X then $\phi^0(x)$ is the identity on X and $\phi^n(x) = \phi(\phi^{n-1}(x))$ for every natural number n and every $x \in X$. Clearly ϕ^n can be considered as a mapping from X into X.

Example. The following example is to show that Theorems 1 and 2 are independent. It meets the requirements of Theorem 2 but not of Theorem 1.

Let X be the collection of all real seqences $\{x_i\}_{i=1}^{\infty}$ with $|x_i| \leqq 2^{-i}$ (this is the Hilbert cube). We consider the usual metric. We define a contraction $\phi : X \to X$ by

$$\phi : \{x_i\}_{i=1}^{\infty} \mapsto \left\{\frac{i}{i+1}\, x_i\right\}_{i=1}^{\infty} .$$

ϕ is clearly a contraction on a compact space but there is no contraction constant $\alpha < 1$ such that $\varrho(\phi(x), \phi(y)) \leqq \alpha\, \varrho(x, y)$, for every x and y in X.

Theorem 3. *Let (X, ϱ) be a metric space, and let ϕ be a continuous function from X into X which satisfies the following properties*:

(i) $\exists x_0 \in X$ *such that* $\{\phi^n(x_0)\}_{n=1}^{\infty}$ *contains a convergent subsequence in* X.

(ii) $\forall x \in X$; $\forall y \in X$ *we have* $\lim_{n\to\infty} \varrho(\phi^n(x), \phi^n(y)) = 0$.

Then the space X contains exactly one fixed point relative to the transformation ϕ.

Proof. Since $\{\phi^n(x_0)\}_{n=1}^{\infty}$ contains a convergent subsequence in (X, ϱ) there exists an infinite subset M of the natural numbers such that $\{\phi^m(x_0) \mid m \in M\}$ is convergent. Let $\hat{x}_0$ be its limit. From the continuity of ϕ it follows that $\{\phi^{m+1}(x_0) \mid m \in M\}$ is convergent with limit $\phi(\hat{x}_0)$. Choose an $\varepsilon > 0$. From condition (ii) it follows that $\exists N_0$ such that for every natural number $n > N_0$ we have

$$\varrho(\phi^n(x_0), \phi^{n+1}(x_0)) < \tfrac{1}{3}\varepsilon .$$

Furthermore there exists an N_1 such that

$$\forall m \in M ; \quad m > N_1 \quad \text{we have} \qquad \varrho(\phi^m(x_0), \hat{x}_0) < \tfrac{1}{3}\varepsilon$$

and

$$\forall m \in M ; \quad m > N_1 \quad \text{we have} \quad \varrho(\phi^{m+1}(x_0), \phi(\hat{x}_0)) < \tfrac{1}{3}\varepsilon .$$

Since M is infinite we conclude that $\varrho(\hat{x}_0, \phi(\hat{x}_0)) < \varepsilon$ for every positive number ε and therefore $\hat{x}_0$ has to be a fixed point of ϕ. Suppose that $\hat{y}_0$ is another fixed point, then $\lim_{n\to\infty} \varrho(\phi^n(\hat{x}_0), \phi^n(\hat{y}_0)) = 0$. Since $\hat{x}_0 = \phi(\hat{x}_0) = \phi^n(\hat{x}_0)$ and $\hat{y}_0 = \phi(\hat{y}_0) = \phi^n(\hat{y}_0)$ for all $n \in \mathbf{N}$ we have $\hat{x}_0 = \hat{y}_0$. Therefore $\hat{x}_0$ is the unique fixed point of the function ϕ.

Theorem 4. *Let X be a Tychonoff space and let ϕ be a continuous mapping from X into X. If there exists a compatible uniform structure $\mathscr{H}$ on X such that*

$\forall x \in X$, $\forall y \in X$, $\forall H \in \mathscr{H}$, $\exists N_0 \in \mathbf{N}$ *such that* $\forall n \in \mathbf{N}$ *with* $n > N_0$ *we have* $(\phi^n(x), \phi^n(y)) \in H$,

then the following conditions are equivalent:

(i) $\exists x_0 \in X$ *and an infinite subset M of* $\mathbf{N}$ *such that* $\{\phi^m(x_0) \mid m \in M\}$ *is a convergent sequence.*

(ii) *The space X contains exactly one fixed point $\hat{x}_\phi$ relative to ϕ.*

(iii) *For every $x \in X$ the sequence $\{\phi^n(x) \mid n \in \mathbf{N}\}$ converges.*

Proof. (i) $\Rightarrow$ (ii). Let $\hat{x}$ be the limit of $\{\phi^m(x_0) \mid m \in M\}$. From the continuity it follows that $\{\phi^{m+1}(x_0) \mid m \in M\}$ converges to $\phi(\hat{x})$. Let H be an arbitrary diagonal neighbourhood in $\mathscr{H}$. Then there exists a $K \in \mathscr{H}$ such that $K = K^{-1}$ and $K \circ K \circ K \subset H$. There exists an $N_0 \in \mathbf{N}$ such that

(a) $(\phi^m(x_0), \phi^{m+1}(x_0)) \in K$ for every $m \in \mathbf{N}$; $m \geqq N_0$.
(b) $(\phi^m(x_0), \hat{x}) \in K$ for every $m \in M$; $m \geqq N_0$.
(c) $(\phi^{m+1}(x_0), \phi(\hat{x})) \in K$ for every $m \in M$; $m \geqq N_0$.

Since M is infinite we can choose m sufficiently large in M and we conclude that

$$(\hat{x}, \phi(\hat{x})) \in K \circ K \circ K \subset H .$$

Since X is a Tychonoff space it follows that $\hat{x} = \phi(\hat{x})$. Suppose that $\hat{y}$ is another fixed point of ϕ, then

$$\forall H \in \mathscr{H} ; \quad \exists N \in \mathbf{N} ; \quad \forall n > N \quad \text{we have} \quad (\phi^n(\hat{y}), \phi^n(\hat{x})) = (\hat{y}, \hat{x}) \in H .$$

This implies that $\hat{y} = \hat{x}$. Therefore $\hat{x}$ is the unique fixed point of ϕ in X.

(ii) $\Rightarrow$ (iii). Let x_0 be the fixed point of ϕ in X and let x be an arbitrary point of X. Let U be a neighbourhood of x_0. Then U contains a neighbourhood of x_0 of the form: $\{y \mid (y, x_0) \in H\}$ for some $H \in \mathscr{H}$. By definition there exists an $N_0 \in \mathbf{N}$ such that for every $n > N_0$ we have $(\phi^n(x), \phi^n(x_0)) \in H$; hence $(\phi^n(x), x_0) \in H$. Therefore $\phi^n(x)$ is eventually in every neighbourhood of x_0, i.e., $\phi^n(x)$ converges to x_0.

(iii) $\Rightarrow$ (i). Obvious.

Proof of Theorem 1. Let x and y be two arbitrary points of X. Then $\varrho(\phi^{n+1}(x), \phi^{n+1}(y)) \leqq \alpha \,.\, \varrho(\phi^n(x), \phi^n(y)) \leqq \alpha^{n+1} \,.\, \varrho(x, y)$. Since $\alpha < 1$ we have $\lim_{n\to\infty} \varrho(\phi^n(x), \phi^n(y)) = 0$.

Moreover, for every $x \in X$ we have

$$\varrho(\phi^n(x), x) \leqq \sum_{i=1}^{n} \varrho(\phi^i(x), \phi^{i-1}(x)) \leqq \sum_{i=1}^{n} \alpha^{i-1} \,.\, \varrho(\phi(x), x) \leqq \frac{1}{1-\alpha}\, \varrho(\phi(x), x) .$$

Therefore, for every k and $l \in \mathbf{N}$, $k \geqq l$ we have

$$\varrho(\phi^k(x), \phi^l(x)) \leqq \alpha^l \,.\, \varrho(\phi^{k-l}(x), x) \leqq \frac{\alpha^l}{1-\alpha} \,.\, \varrho(\phi(x), x) .$$

This implies that $\{\phi^n(x) \mid n \in \mathbf{N}\}$ is a Cauchy sequence and from the completeness of X it follows that its limit exists. We conclude that $\{\phi^n(x) \mid n \in \mathbf{N}\}$ satisfies the condition (i) in Theorem 3 and Theorem 1 follows from Theorem 3.

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References

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