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# ON SOME PROBLEMS OF LOCAL APPROXIMABILITY IN COMPACT SPACES

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## 1. Introduction.

In this paper we study some problems concerning $T_2$ compact spaces. The main problem, not yet solved, is the following:

Problem 1. Is a compact separable and accessible Hausdorff space $E$ first countable?

We give three examples showing that the conjecture is not true if the compact separable and accessible space is not Hausdorff or if the separable and accessible Hausdorff space fails to be compact or finally, if the compact Hausdorff space is neither separable nor accessible.

Another question arises if we require such a space to be Fréchet or sequential. In this paper it is proved that a $T_2$ compact, sequentially compact space is sequential provided that the topology induced by the sequential closure is Lindelöf.

Another interesting result deduced from a proposition of Mrówka is given, concerning the points of countable character in such spaces.

**2.** Let $(E, \tau)$ be a topological space where $E$ is a set and $\tau$ a topological structure. Let us denote by $(E, \lambda)$ a space where $\lambda$ denotes the convergence structure on $E$ (in the sense of Dolcher [5], or Novák [9]). Let $(E, \mu)$ be a closure space, i.e., a space $E$ where a closure $\hat{A}$ is defined such that it does not necessarily satisfy the axiom of the closed closure $\hat{\hat{A}} = \hat{A}$. Let us denote by $T, L, M, N$ the following functors $T$: $: (E, \lambda) \to (E, \tau)$; $L: (E, \tau) \to (E, \lambda)$; $M: (E, \lambda) \to (E, \mu)$; $N: (E, \mu) \to (E, \tau)$ (see [5], [7]).

**Definition.** A topological space $(E, \tau)$ is said to be *sequential* iff $TL(\tau) = \tau$; it is said to be *Fréchet* iff $ML(\tau) = \tau$.

Remark. These definitions are equivalent to the well-known ones; they are expressed in terms of categories and functors.

It is well known that $NM = T$.

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**Definition.** Let $(E, \tau)$ be a separable space with a countable subset $D$ dense in $E$; we shall say that $E$ is *accessible from* $D$ with the topology $\tau$ iff for every point $x$ of $E$ there is a one-to-one sequence of points of $D$ converging to $x$.

**Definition.** A space $E$ is said to be *sequentially compact* iff every sequence of points of $E$ contains a convergent subsequence.

**3.** The main problem is to find sufficient conditions for a $T_2$ compact space to be first countable. The simple hypothesis that $(E, \tau)$ is a $T_2$ compact separable space is not sufficient. This is shown by the example of $I^I$ with the product topology where $I = [0, 1]$.

Using a result of Mrówka [8]: "Under continuum hypothesis a $T_2$ compact space of cardinality $\mathfrak{c}$ has a subset of the same cardinality in which the first countability axiom holds", we can immediately prove the following propositions (continuum hypothesis is needed for Propositions 2, 3, 4):

**Proposition 1.** *A separable and accessible topological space* $(E, \tau)$ *has cardinality at most* $\mathfrak{c}$.

**Proposition 2.** *A* $T_2$ *compact separable accessible and homogeneous topological space* $(E, \tau)$ *is first countable.*

The proof follows immediately from Proposition 1 and from the result of Mrówka.

**Proposition 3.** *A* $T_2$ *compact separable and accessible topological group is metrizable.*

The proof follows from Proposition 2 and from a well-known criterion of the metrizability of topological groups with countable local basis (see [3]).

**Proposition 4.** *A* $T_2$ *compact topological space* $(E, \tau)$ *with cardinality at most* $\mathfrak{c}$ *(in particular, separable and accessible) has a dense subset* $D$ *every point of which satisfies the first axiom of countability.*

Proof. Let $x$ be a point of $E$ and $U$ an open neighbourhood of $x$. $E$ is completely regular and therefore there exists a continuous function $f$ on the set $E$ into $[0, 1]$ such that $f(x) = 0$ and $f(\mathscr{C}U) = 1$ where $\mathscr{C}U$ is the complement of $U$ in $E$. Let $C = f^{-1}(0)$. $C$ is a subset of $U$ and it is $G_\delta$. Since $C$ is closed in $E$, it is $T_2$ compact. By the result of Mrówka there is $y \in C$ which has a countable basis in the topology of $C$. But $C$ is a $G_\delta$ and because of the transitivity of the character of a point (see [1], [2], [4]) $y$ has a countable basis also in $E$.

The space $E$ in Problem 1 is (1) compact; (2) Hausdorff; (3) separable; (4) accessible. We give now an example of a space satisfying (2), (3), (4) but not (1) which is not first countable.

Example 1. Let $E$ be a square: $[0, 1] \times [0, 1]$. The neighbourhoods of points $(a, b) \neq (1, 0)$ are the same as in the usual topology of $R^2$ restricted on $E$. The basis of the neighbourhoods of $(1, 0)$ is the following family of sets:

$$V_{\bar{y},g}((1, 0)) = \{(x, y) : g(y) < x \leqq 1, y < \bar{y}\},$$

where $g$ is a continuous function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$, $g(x) \neq 0$, $\forall x \neq 0$, and $\bar{y}$ a point such that $0 < \bar{y} \leqq 1$.

Now we construct a space satisfying (1), (3), (4) but not (2) which is not first countable.

Example 2. Let $E$ be the set given by the union of the closed interval $I = [0, 1]$ and a point $x_0 \notin I$. The topology $\tau$ is the following: the neighbourhoods of the points of $I$ are the same as in the usual topology for the interval. A basis of neighbourhoods of $x_0$ is given in the following way:

$$V_{x_1,\ldots,x_n}(x_0) = [I - \{x_1, x_2, \ldots, x_n\}] \cup \{x_0\}, \; x_i \in I \quad (i = 1, 2, \ldots, n).$$

We finally give an example of a topological space satisfying (1), (2) but neither (3) nor (4) which is not first countable.

Example 3. Let $E$ be the set of all transfinite ordinals less than or equal to $\omega_1$ where $\omega_1$ is the first uncountable ordinal. Obviously, the space $E$ is not first countable.

**4.** Problem 1 can be decomposed in a number of problems, every one of which can be interesting. For example, to find sufficient conditions for a compact topological space to be sequential or to be Fréchet. In fact, it is well-known that every first countable space is Fréchet and every Fréchet space is sequential ([6]). We have obtained the following result:

**Lemma 1.** *In a Hausdorff topological space $(E, \tau)$ the convergent sequences are the same as in the space $(E, TL(\tau))$. In particular, if $(E, \tau)$ is sequentially compact, so is $(E, TL(\tau))$.*

The proof is obvious.

**Proposition 5.** *Let $(E, \tau)$ be a $T_2$ compact, sequentially compact space. Let $(E, TL(\tau))$ be Lindelöf. Then $(E, \tau)$ is a sequential space.*

Proof. Let us consider the topological space $(E, \tau)$ and suppose that it is not sequential. This means $TL(\tau) \neq \tau$ and consequently $TL(\tau)$ is strictly finer than $\tau$. This means that in the $TL(\tau)$ topology there are open sets not open in $\tau$. Therefore

$(E, TL(\tau))$ cannot be a compact space. But we know that $(E, \tau)$ is sequentially compact and therefore every sequence in $(E, TL(\tau))$ has a convergent subsequence (by Lemma 1). Now $(E, TL(\tau))$ is not compact, therefore there exists an open covering of $E$ that has no finite subcovering. But this covering must have a countable subcovering, because $(E, TL(\tau))$ is Lindelöf. Hence there exists a sequence in $(E, TL(\tau))$ without convergent subsequences and this contradicts the hypothesis that $(E, \tau)$ is sequentially compact. Therefore $(E, \tau)$ is sequential.

Remark. A compact and sequentially compact space need not be sequential. This is shown by Example 3.

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