Toposym 3

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In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 471--473.

Persistent URL: http://dml.cz/dmlcz/700742

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SPACES WITH REGULAR G_δ-DIAGONALS

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The proofs that are omitted in this note will appear in [6]. Recall that a subset H of the space X is a regular G_δ-set if there is a sequence $\{U_n\}$ of open sets containing H such that $H = \bigcap_{i=1}^{\infty} U_i = \bigcap_{i=1}^{\infty} U_i^-$. A space X has a (regular) G_δ-diagonal if $\{(x, x) : x \in X\}$ is a (regular) G_δ-set in $X \times X$. In [2], Ceder obtains the following characterization of spaces with G_δ-diagonals:

Theorem 1. *X has a G_δ-diagonal if and only if there is a sequence $\{\mathscr{G}_n\}$ of open covers of X such that if $x \in X$, then $x = \bigcap_{i=1}^{\infty} \operatorname{st}(x, \mathscr{G}_i)$.*

We have a comparable characterization of spaces with regular G_δ-diagonals:

Theorem 2. *X has a regular G_δ-diagonal if and only if there is a sequence $\{\mathscr{G}_n\}$ of open covers of X such that if x and y are distinct points of X, then there are an integer n and open sets U and V containing x and y respectively such that no member of $\mathscr{G}_n$ intersects both U and V.*

From Theorems 1 and 2, it is quite easy to see that any paracompact T_2-space with a G_δ-diagonal has a regular G_δ-diagonal. Also, it is a corollary to Theorem 2 that any space with a regular G_δ-diagonal is Hausdorff.

A development $\{\mathscr{G}_n\}$ for the space X is said to satisfy the 3-link property if it is true that if p and q are distinct points, then there is an integer n such that no member of $\mathscr{G}_n$ intersects both $\operatorname{st}(p, \mathscr{G}_n)$ and $\operatorname{st}(q, \mathscr{G}_n)$ (Heath [3]). According to Borges [1], a space X is a $w\Delta$-space if there is a sequence $\{\mathscr{G}_n\}$ of open covers of X such that if x is a point and, for each n, x_n is a point of $\operatorname{st}(x, \mathscr{G}_n)$, then the sequence $\{x_n\}$ has a cluster point.

Theorem 3. *Let X be a topological space. Then the following conditions are equivalent:*

(a) *X admits a development satisfying the 3-link property.*

(b) *X is a $w\Delta$-space with a regular G_δ-diagonal.*

(c) *There is a semi-metric d on X such that:*

(i) *If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences both converging to* x, *then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

(ii) *If* x *and* y *are distinct points of* X *and* $\{x_n\}$ *and* $\{y_n\}$ *are sequences converging to* x *and* y *respectively, then there are integers* N *and* M *such that if* $n > N$, *then* $d(x_n, y_n) > 1/M$.

According to Morita [5], a space X is an M-space if there is a normal sequence $\{\mathscr{G}_n\}$ of open covers of X such that if x is a point and, for each n, x_n is a point of $\operatorname{st}(x, \mathscr{G}_n)$, then the sequence $\{x_n\}$ has a cluster point.

Theorem 4. *If* X *is a topological space, then the following conditions are equivalent*:

(a) X *is metrizable.*

(b) X *is a* T_1-M-*space such that* X^2 *is perfectly normal.*

(c) X *is an* M-*space with a regular* G_δ-*diagonal.*

(d) X *is a* T_1-M-*space such that* X^3 *is hereditarily normal.*

(e) X *is a* T_1-M-*space such that* X^3 *is hereditarily countably paracompact.*

(f) X *is an* M-*space that admits a one-to-one continuous function onto a metric space.*

Finally, in [1], Borges shows that if X is paracompact, locally connected and locally peripherally compact, then X is metrizable if and only if X has a G_δ-diagonal. Borges' result follows as a corollary to the following theorem:

Theorem 5. *If* X *is locally connected and locally peripherally compact, then* X *is metrizable if and only if* X *has a regular* G_δ-*diagonal.*

Proof. Let $\{\mathscr{U}_n\}$ be a sequence of open covers of X such that each member of $\mathscr{U}_n$ is connected and such that if p and q are distinct points, then there are open sets U and V containing p and q respectively and an integer n such that no member of $\mathscr{U}_n$ intersects both $\operatorname{st}(p, \mathscr{U}_n)$ and $\operatorname{st}(q, \mathscr{U}_n)$. We will first show that $\{\mathscr{U}_n\}$ is a development for X. To this end, let $x \in X$ and let U be an open set containing x. There is an open set V with a compact boundary such that $x \in V \subset U$. Suppose that, for each n, there is a member, say g_n, of $\mathscr{U}_n$ that contains x and intersects $X - V$. Then, since each g_n is connected, there is a point x_n of the boundary of V that is in g_n. Since the boundary of V is compact, the sequence $\{x_n\}$ has a cluster point, say x_0. It follows that $x_0 \in \bigcap_{n=1}^{\infty} \operatorname{cl}(\operatorname{st}(x, \mathscr{U}_n))$ which is a contradiction. It follows that X is developable. By Theorem 3, there is a development $\{\mathscr{G}_n\}$ for X that satisfies the 3-link property. Since X is locally connected, we may assume that, for each n, the members of $\mathscr{G}_n$ are connected. Let x denote a point of X and let U be an open set containing x. We will show that there is an integer n such that if $g \in \mathscr{G}_n$ and $g \cap \operatorname{st}(x, \mathscr{G}_n) \neq \emptyset$, then $g \subset U$. It will then follow that X is metrizable by Moore's Metrization Theorem

[4]. To this end, let V be an open subset of U containing x with compact boundary. Suppose that, for each n, there are members U_n and V_n of $\mathcal{G}_n$ such that $x \in U_n$, $U_n \cap \cap V_n \neq \emptyset$, and $(U_n \cap V_n) \cap (X - V) \neq \emptyset$. Since, for each n, $U_n \cup V_n$ is connected, there is a point x_n of $U_n \cup V_n$ in the boundary of V. Since the boundary of V is compact, there is a cluster point, x_0, of $\{x_n\}$. But it follows that, for each n, there is a member of $\mathcal{G}_n$ that intersects both of st $(x, \mathcal{G}_n)$ and st $(x_0, \mathcal{G}_n)$ which is a contradiction, from which the theorem follows.

References

[1] *C. J. R. Borges:* On metrizability of topological spaces. Canad. J. Math. *20* (1968), 795—803.
[2] *J. G. Ceder:* Some generalizations of metric spaces. Pacific J. Math. *11* (1961), 105—125.
[3] *R. W. Heath:* Metrizability, compactness and paracompactness in Moore spaces. Notices Amer. Math. Soc. *10* (1963), 105.
[4] *R. L. Moore:* A set of axioms for plane analysis situs. Fund. Math. *25* (1935), 13—28.
[5] *K. Morita:* Products of normal spaces with metric spaces. Math. Ann. *154* (1964), 365—382.
[6] *P. Zenor:* On spaces with regular G_δ-diagonals. Pacific J. Math. (to appear).

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