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RELATIONS BETWEEN $\mathfrak{B}$-COMPLETENESS AND m-PARACOMPACTNESS

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This communication studies the relations between m-paracompactness and $\mathfrak{B}$-completeness for some classes $\mathfrak{B}$ of directed sets. Most of the results are similar to [4], but the proofs are new and more simple. We shall show that m-paracompactness implies $\mathfrak{N}_m$-completeness in completely regular spaces. Equivalence between these two notions does not hold in general, but it takes place when the space is supposed to be a generalized order closure space (Theorem 2). The rest of the paper is devoted to the closed relations in $\mathfrak{B}$-spaces.

$\mathfrak{B}$ denotes any class of directed sets, $\mathfrak{N}$ the class of monotone ordered sets, $\mathfrak{N}_m = \{\langle D, \prec\rangle \in \mathfrak{N} \mid \text{card } D \leqq m\}$; a net is called $\mathfrak{B}$-net iff its domain belongs to $\mathfrak{B}$. A net N is called *remarkable* in $\mathscr{P} = \langle P, u\rangle$ iff $f \circ N$ converges in $I = [\![0, 1]\!]$ for each $f \in C = \mathscr{C}(\mathscr{P}, I)$, equivalently, if the range of N is in $\mathscr{P}$ and N converges in $\beta\langle P, \tilde{u}\rangle$ ($\tilde{u}$ being the completely regular modification of u). A closure space $\mathscr{P}$ is called *$\mathfrak{B}$-complete* iff every $\mathfrak{B}$-net remarkable in $\mathscr{P}$ converges in $\mathscr{P}$ (contrary to [3], we do not suppose in this definition that $\mathscr{P}$ is a $\mathfrak{B}$-space).

Theorem 1. *For every cardinal number* m, *every* m*-paracompact completely regular space is* $\mathfrak{N}_m$*-complete.*

Proof. Let $\mathscr{P} = \langle P, u\rangle$ be an m-paracompact completely regular space. It is sufficient to prove that each non-convergent $\mathfrak{N}_m$-net is not remarkable. Let N be such a net. Without loss of generality we may assume that N is one-to-one and such that $\mathbf{D}N = \langle \alpha, \in\rangle$ where $\alpha \leqq m$ is a regular ordinal.

Let us denote $U_\eta = P - uN[\![\eta, \rightarrow]\!]$ for each $\eta < \alpha$. Then $\mathscr{U} = \{U_\eta \mid \eta < \alpha\}$ is an increasing open cover of $\mathscr{P}$ (this follows easily from the fact that N does not converge) and card $\mathscr{U} \leqq m$. Therefore, by [2], there exists a locally finite open cover $\mathscr{Z}$ such that $\{uZ \mid Z \in \mathscr{Z}\}$ refines $\mathscr{U}$.

We shall construct an increasing map $d : \alpha \rightarrow \alpha$ and a disjoint and locally finite family $\{V_\xi \mid \xi < \alpha\}$ of open neighbourhoods of points $Nd\xi$.

Transfinite induction. $\eta = 0 : d0 = 0$, $V_0 = U_0 \cap Z_0$, where $Z_0 \in \mathscr{Z}$, $Nd0 \in Z_0$ and U_0 is an open neighbourhood of $Nd0$, which intersects only finitely many members of $\mathscr{Z}$.

Let $0 < \eta < \alpha$ and suppose that $d\xi$, V_ξ and Z_ξ have been defined for all $\xi < \eta$. Since α is regular and since $u[\mathscr{Z}]$ refines $\mathscr{U}$, then exists a $\lambda < \alpha$ with $N\lambda \notin$

$\notin \mathsf{u} \bigcup\{Z_\xi \mid \xi < \eta\}$. There exists a $Z_\eta \in \mathscr{Z}$ with $N\lambda \in Z_\eta$ and an open neighbourhood U_η of the point $N\lambda$, which interesects only finitely many members of $\mathscr{Z}$. For the induction step it remains to define:

$$\lambda = d\eta\,, \quad V_\eta = U_\eta \cap Z_\eta - \mathsf{u} \bigcup\{Z_\xi \mid \xi < \eta\}\,.$$

For each $\xi < \alpha$ we can choose a $g_\xi \in \mathsf{C}$ such that $g_\xi[P - V_\xi] = \{0\}$, $g_\xi N d_\xi = 1$. If S and $\alpha - S$ are cofinal subsets of α, the function $g = \sum\{g_\xi \mid \xi \in S\}$ is correctly defined, continuous and belongs to C. Moreover, gN is equal to 1 and 0 respectively on cofinal subsets $d[S]$ and $d[\alpha - S]$ of α, hence gN does not converge in I. Thus N is not remarkable, which completes the proof.

Proposition 1. *If some proper maximal filter $\langle j, \supset \rangle$ of open sets of a completely regular space $\mathscr{P}$ belongs to $\mathfrak{B}$ (or is a quotient of some element of $\mathfrak{B}$), then $\mathscr{P}$ is not $\mathfrak{B}$-complete.*

The proof is obvious: A net $\{N_U \mid U \in j\}$, where $N_U \in U$, converges to j in $\beta\mathscr{P}$.

Therefore, for separated spaces and sufficiently large $\mathfrak{M}$, *the $\mathfrak{M}$-completeness coincides with compactness (and hence with $\mathfrak{N}$-compactness)*. On the other hand, any infinite discrete space is $\mathfrak{N}$-complete.

Proposition 2. *Every product of completely regular paracompact spaces is $\mathfrak{N}$-complete.*

To prove Proposition 2 notice that any $\mathfrak{B}$-completeness is a productive property, and apply Theorem 1.

Proposition 2 enables us to show that the only if part in Theorem 1 cannot be true in general: Any non-paracompact product of paracompact spaces (e.g. Sorgenfrey's square) serves as an example of a non-paracompact $\mathfrak{N}$-complete space.

Theorem 2. *A generalized order closure space is $\mathfrak{N}_{\mathsf{m}}$-complete if and only if it is m-paracompact.*

Proof. Assume $\mathscr{P} = \langle P, \mathsf{u} \rangle$ is not m-paracompact. Without loss of generality we may assume that there exist an open-closed interval-like subspace $\mathscr{Q}'$ of $\mathscr{P}$, a point $z \in Q'$, a regular ordinal $\gamma \leqq m$ and an increasing net $N = \{N\xi \mid \xi \in \gamma\}$ such that the open cover $\mathscr{W} = \{]\!]z, N\xi[\![\mid \xi \in \gamma\}$ of $Q = Q' \cap]\!]z, \rightarrow[\![$ is not uniformizable. N does not converge. The existence of such $\mathscr{W}$ follows by [2]; for the details, see [4].

Suppose $f \circ N$ does not converge in I for some $f : P \to [\![0, 1]\!]$. Then $f \circ N$ is frequently in two sets A and B separated in I and we can choose an increasing map $h : \gamma \to \gamma$ such that $N \circ h$ lies alternately in $f^{-1}[A]$ and $f^{-1}[B]$. For each $t \in Q$ we can define the minimal $m_t \in \alpha$ such that $t = Nhm_t$; then $]\!]z, Nh(m_t + 1)[\![$ is a neighbourhood of t. Since $\mathscr{W}$ is not uniformizable, there exists (see [1], p. 435) $R \subset Q$ and $y \in \mathsf{u}R - \bigcup\{]\!]z, Nh(m_t + 1)[\![\mid t \in R\}$. We can prove that $y \in \mathsf{u}f^{-1}[B] \cap$

$\cap$ u$f^{-1}[A]$, hence f is not continuous. Therefore, N is remarkable and $\mathscr{P}$ is not $\mathfrak{N}_m$-complete.

Let $\mathscr{X}$ be a $\mathfrak{B}$-compact space (which means that every $\mathfrak{B}$-net ranging in X has an accumulation point in $\mathscr{X}$), *let the Cartesian product $\mathscr{X} \times \mathscr{X}$ be a $\mathfrak{B}$-space* (i.e., its closure u is determined by a convergence of $\mathfrak{B}$-nets). *Then the composition of any two (or finitely many) closed relations is a closed relation* (i.e., $(\mathrm{u}R = R \subset X \times X \;\&\; \mathrm{u}S = S \subset X \times X) \Rightarrow \mathrm{u}(R \circ S) = R \circ S$).

The problem arises whether the product of two closed relations is closed, provided that $\mathscr{X}$ is a $\mathfrak{B}$-compact $\mathfrak{B}$-space for some $\mathfrak{B}$.

Let $\mathscr{X}$ be separated, let D be discrete in $\mathscr{X}$, let M be a net converging to x in $\mathscr{X}$, let N be a net converging to $y \neq x$ in $\mathscr{X}$ such that $\aleph_0 \leqq \text{card } \mathbf{E}M = \text{card } \mathbf{E}N \leqq \text{card } D$. Then there exist closed equivalences R and S on $\mathscr{X}$ such that neither $R \circ S$ nor its transitive envelope is closed.

References

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