ON TOTAL ORDERINGS IN TOPOLOGY

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In an orderable topological space the following cardinal invariants coincide: local neighborhood character, pseudocharacter, sequential character, and Fréchet character (definitions in § 2). This paper considers the extent to which this result can be extended to classes of spaces satisfying some related total order conditions (list in § 1) which have appeared in the literature. The implications which hold between these conditions (there are two more in § 3) and examples to show their independence are given.

1. Some total order conditions from the literature

We begin by listing the order conditions to be considered. Unfortunately the terminology in the literature is not uniform. No attempt is made here to give a complete list of references; those given are either the earliest known to the author or those which are most useful for the present work.

1.1. Definition. Let (X, t) be a topological space.

1) t is *orderable* if there is a total ordering of X for which t is the order topology.

2) t is *locally orderable* if each point has an open neighborhood on which the relative topology is orderable [10].

3) and 4) Assume t is T_1. If $A \subset X$, the "chain net closure" of A consists of all limits of chain nets in A. (A chain net is a net whose directed set is totally ordered.) If the chain net closure operator coincides with the t-closure operator, t is called a *strong chain net topology.* If iteration of the chain net closure operator yields the t-closure operator, t is called a *chain net topology,* [4] and [5].

5) t is called a *nested neighborhood topology* if it is T_1 and each point has a local neighborhood base that is totally ordered by inclusion [2].

6) t is called a *generalized orderable topology* (GO topology) if there is a total ordering for X such that: t is larger (finer) than the order topology, and each point has a t-neighborhood base consisting of (possibly degenerate) order intervals [3].

7) t is called *weakly orderable* if there is a total ordering for X for which order-open rays are topologically open.

We will sometimes find it convenient to refer to these conditions by their initial letters. Two other related conditions, which do not seem to have been studied, are given in § 3.

1.2. Remark. A locally orderable space is T_1 but need not be T_2; in fact sequential limits need not be unique. (The space consisting of a sequence of isolated points converging to two distinct limits is locally orderable.) Thus all spaces considered in this paper are at least T_1, because T_2 is implied by each of the other conditions for which no separation axiom was assumed.

1.3. Theorem. *The above conditions are related as follows*: O → LO → SCN → → CN; O → GO → WO; GO → SCN; NN → SCN. *No other (non-trivial) implications are valid, even for T_2 spaces.*

Proof. The implications are all easy and/or known. Examples 1 through 5 justify the assertion that there are no other implications.

Some terminology and notation will be useful for the examples. **W** (**W***) denotes the set of all ordinals $<\omega_1$ ($\leqq\omega_1$). **N** is the set of positive integers; $\mathbf{N}^* = \mathbf{N} \cup \{\omega_0\}$. If X and Y are totally ordered sets, there are two total orderings on $X \times Y$ which will be used. The order topology induced by ordering by first (last) differences will be called the *left* (*right*) *lexicographic order topology*.

Example 1. The subset of $\mathbf{W}^* \times \mathbf{N}^*$ consisting of the top edge and the right edge, $(\{\omega_1\} \times \mathbf{N}^*) \cup (\mathbf{W}^* \times \{\omega_0\})$, is orderable but not NN.

Example 2. A circle is locally orderable and NN but not WO. (Note however that this topology can be obtained as the intersection of two orderable topologies.)

Example 3. The Sorgenfrey line is a GO space and NN but not locally orderable. (Proof as in [7].)

Example 4. We construct a WO space that is not a chain net space. Let $X = (\mathbf{W}\times\mathbf{N}) \cup \{p\}$, where $p = (\omega_1, \omega_0)$. Let t_1 (t_2) be the left (right) lexicographic order topology on the product with p as last point, and put $t = t_1 \vee t_2$ (join in the lattice of all topologies on X). The join of orderable topologies is a fortiori WO. The fact that t is not CN can be seen by observing that t-neighborhoods of p are the same as relative product neighborhoods from $\mathbf{W}^* \times \mathbf{N}^*$.

2. On extending cardinal invariant properties of orderable spaces

For a topological space (X, t) there are four cardinal invariants to be considered here. For simplicity we consider only infinite cardinals. The *local neighborhood character* χX is the least cardinal $\mathfrak{m}$ such that each point has a local neighborhood base of cardinality not greater than $\mathfrak{m}$. The *pseudocharacter* ψX is the least $\mathfrak{m}$ such that each point can be expressed as the intersection of at most $\mathfrak{m}$ open sets.

The other invariants arise from two interpretations of the question: what is the smallest $\mathfrak{m}$ such that one can recover t from the t-convergent $\mathfrak{m}$-nets? (An $\mathfrak{m}$-*net* is a net whose directed set has cardinality $\leqq \mathfrak{m}$. The cardinality of the directed set is also called the *size* of the net.) The $\mathfrak{m}$-closure of a subset A consists of all limits of t-convergent $\mathfrak{m}$-nets in A. The least $\mathfrak{m}$ for which the $\mathfrak{m}$-closure operator is the t-closure operator is called the *Fréchet character* of X, ϕX. The least $\mathfrak{m}$ for which iteration of the $\mathfrak{m}$-closure operator yields the t-closure operator is called the *sequential character*, σX [10]. The spaces for which $\sigma X = \aleph_0$ ($\phi X = \aleph_0$) are the sequential (Fréchet) spaces.

In general $\sigma X \leqq \phi X \leqq \chi X$ and $\psi X \leqq \chi X$, and all of the inequalities can be strict. However, for orderable spaces all four invariants coincide. To what extent does this result extend to spaces satisfying the total order conditions in § 1? The complete answer:

2.1. Theorem. (i) *If* (X, t) *is any one of the following*:

GO *space,*

nested neighborhood space,

locally orderable space,

then $\sigma X = \phi X = \chi X = \psi X$.

(ii) *If* (X, t) *is a strong chain net space then* $\sigma X = \phi X$. *In other words, if the t-closure operator can be obtained in one step using totally ordered directed sets of minimal size, then allowing iteration will not enable one to use smaller directed sets, totally ordered or otherwise.*

(iii) *No other such equalities hold in any of the classes of spaces in* § 1.

Proof. Cases of (i) were proved in [9] and [10]; the other proofs are similar. Examples 5 through 8 justify (iii).

Part of the next Corollary was obtained independently by C. Aull [1].

2.2. Corollary. *The notions of sequential space, Fréchet space and first countable space are equivalent in the following classes of spaces*: GO *spaces, locally orderable spaces, and nested neighborhood spaces.*

Proof. This is the countable case of (i).

We now give the examples to show that no other cardinal equalities are possible. We begin by showing that three well known examples can be obtained as enlargements (more open sets) of the same order topology. Let $M = (\mathbf{N} \times \mathbf{N}) \cup \{p\}$

and u = the order topology on M from the following ordering: left lexicographic order on $\mathbf{N} \times \mathbf{N}$ with p as last element.

Example 5. Let t_1 be the sequential topology on M generated by the following u-convergent sequences: $((k, 1)) \to p$ and $((k, n)) \underset{n}{\to} (k + 1, 1)$ for each k (i.e., the bottom row converges and each column converges). Then $t_1 \supset u$ and (M, t_1) is a WO space which is CN but not SCN and for which $\psi = \sigma = \aleph_0$, $\sigma < \phi$, and $\psi < \chi$.

Example 6. Let t_2 be the Fréchet topology on M generated by the u-convergent rows: for each n $((k, n)) \underset{k}{\to} p$. (M, t_2) is a WO and SCN space for which $\psi = \sigma = \phi = \aleph_0$, but $\phi < \chi$.

Example 7. Let t_3 be the topology on M in which all points except p are isolated and a set containing p is open if and only if it contains all but finitely many points in all but finitely many columns. Thus $t_3 \supset u$ and (M, t_3) is a WO space in which $\aleph_0 = \psi < \chi$ and $\psi < \sigma = \phi$.

Example 8. The one point compactification of an uncountable discrete space provides an example of a SCN space in which $\aleph_0 = \sigma = \phi < \chi = \psi$.

The next theorem describes product spaces which satisfy cardinal equalities. It is a slight extension of [10, Theorem 3.2]. The method of proof is the same.

2.3. Theorem. *Let X be the product of the family $\{X_i : i \in I\}$, where each X_i is a T_1 space with at least two points.*

(i) *If $|I| \geqq \chi X_i$ for each i, then $\sigma X = \phi X = \chi X = \psi X$.*

(ii) *If for each i there is a j such that $\psi X_j = \chi X_j \geqq \chi X_i$ then $\psi X = \chi X$. Corresponding statements with ψ replaced be either σ or ϕ are also valid.*

3. Lattice operations on orderable topologies.

There are two other order conditions for a topological space (X, t) which are related to those in § 1:

(I) t can be expressed in the form $t_1 \cap t_2$ where each t_i is an orderable topology.

(J) t can be expressed as the join (in the lattice of all topologies on X) of two orderable topologies.

3.1. Proposition. *The lattice conditions are related to the conditions in* § 1 *as follows*: O $\to$ (I) $\to$ CN *and* O $\to$ (J) $\to$ WO.

There are two examples which I do not have: a GO space which is not (I) [1]), and a GO space which is not (J). Aside from these two open problems, no other implications among the nine conditions are possible; examples 4, 7, 9, and 10 complete the justification. The implications in Theorem 1.3 and Proposition 3.1 are summarized in Figure 1.

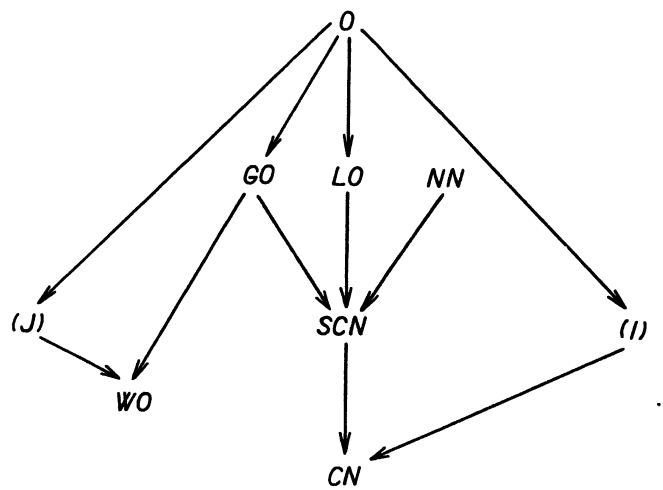

Figure 1.

Example 9. The cross topology on $\mathbf{R} \times \mathbf{R}$ of J. Novák [11] (or see [8]) is the intersection of the left and right lexicographic order topologies. ($\mathbf{R}$ denotes the real line.) This is known to be not a strong chain net topology and can be shown to be not WO (by a connectedness argument [6]).

Example 10. The example in Remark 1.2 satisfies (I), but, if the sequence converges to three points instead of two, we have an example of a LO and NN space which is not (I).

Note added in proof.

The results in § 2 on equality of cardinal functions can be extended to include tightness. More precisely, tightness of X can be added to the equalities in the conclusions of 2.1 Theorem (i) and (ii) and 2.3 Theorem (i). A corresponding statement of 2.3 Theorem (ii) in which tightness of $X = \chi X$ is also valid.

The tightness of X is the least $\mathfrak{m}$ with the property: If A is a subset of X and x is a limit point of A, then x is a limit point of B for some subset B of A with $|B| \leqq \mathfrak{m}$.

[1]) Example 11. A GO space which is not (I) can be obtained by enlarging the usual topology for the real line so as to make the irrationals discrete.

In general tightness of $X \leqq \sigma X$. For more on the relationship of tightness to the other cardinal functions, see the paper of A. V. Arhangelskij in these proceedings.

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