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On profinite universal algebras

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ON PROFINITE UNIVERSAL ALGEBRAS

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This paper deals with a certain part of the somewhat uncharted terrain of universal topological algebra. Singling out profiniteness for study was mainly motivated by the fact that it provides a strong tie to the properties of the underlying algebras so that universal algebraic conditions on the latter could readily be expected to have interesting consequences.

The terminology used here follows, in the main, Grätzer [5] for universal algebra, Mitchell [11] for category theory, and Bourbaki [2] for topology; in particular compactness includes Hausdorffness. The presentation does not include any proofs, beyond the occasional hint; the omitted details are expected to appear elsewhere.

Some of the general material in the first section is contained in, or related to, the doctoral dissertation of my student D. E. Eastman [3] whose interest in universal topological algebra did a great deal to stimulate my own.[1])

1. Compact and pro-C algebras. The topological algebras to be discussed here are particular compact algebras; we therefore begin with some general remarks about the latter.

A topological (universal) algebra is an object $A = (X, (f_\alpha)_{\alpha\in I}, \mathfrak{O})$ where X is a set, $(f_\alpha)_{\alpha\in I}$ a family of maps f_α: $X^{n_\alpha} \to X$, n_α the *arity* of the *operation* f_α, and $\mathfrak{O}$ a topology on X such that each f_α maps the product space $(X, \mathfrak{O})^{n_\alpha}$ *continuously* to the space $(X, \mathfrak{O})$. For such A, $(X, (f_\alpha)_{\alpha\in I})$ is called the *underlying algebra*, and $(X, \mathfrak{O})$ the *underlying space*, of the topological algebra A. Between algebras of the same arity type $\tau = (n_\alpha)_{\alpha\in I}$ one has the familiar notion of homomorphism; when speaking of *topological* algebras, homomorphisms are, unless specified otherwise, understood to be homomorphisms between the underlying algebras which, in addition, map the underlying spaces *continuously*. Clearly, any given class of topological algebras is then the class of objects of a category, the maps (=morphisms) being the homomorphisms between the members of the class; we refer to such categories usually by merely naming their objects.

The most useful classes of topological algebras appear to be those defined by

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specifying a particular algebraic property for the underlying algebras and a topological one for the underlying spaces, e.g. metrizable topological groups, or locally compact topological rings. Of especial significance in the present context are the classes **KA** of topological algebras whose underlying spaces are *compact* and whose underlying algebras belong to a given *equational class* **A** (of algebras) such as, for instance the class **G** of all groups, **Ab** of all abelian groups, R **Mod** of all (left) modules over a given ring R, or **Ann** of all commutative rings with unit.

Also associated with an equational class **A** one has the class **FA** of all finite algebras belonging to **A**; since every finite algebra becomes a compact topological algebra when given the discrete topology of its underlying set, provided all its operations have finite arity[2]), we shall (par abus de langage) also regard **FA** as a subclass of **KA** in that case. Finally, there is the subclass $\mathbf{K_0A}$ of **KA** consisting of all those $A \in \mathbf{KA}$ whose underlying spaces are *zero-dimensional*, and one has $\mathbf{FA} \subseteq \mathbf{K_0A}$ for finite arities.

For any equational class **A**, **KA** as a category has *products*, namely the obvious cartesian ones, and any *closed topological subalgebra* B of an $A \in \mathbf{KA}$, i.e. a topological algebra whose underlying algebra is a subalgebra, and whose underlying space is a *closed* subspace, of that of A, again belongs to **KA**. The latter, together with the fact that the set of points on which two homomorphisms $f, g\colon A \to C$ in **KA** coincide determines a closed subalgebra of A, provides **KA** with equalizers. It follows that **KA** is a complete category.

In addition, any closed congruence Θ on an $A \in \mathbf{KA}$, i.e. a congruence on the underlying algebra of A which is a closed subset of $A \times A$, determines in the usual way operations and a topology derived from those of A on the associated quotient set which make up a topological quotient algebra A/Θ [3]), evidently belonging to **KA**. The existence of these quotients in **KA** readily provides coequalizers. Further, one shows **KA** also has coproducts, and hence it is a cocomplete category. Finally, the functors from **KA** to the category **Ens** of sets by passing to underlying sets, and to the category of, say, completely regular Hausdorff spaces by passing to underlying spaces, have left adjoints.

The latter assertions can be proved directly via a number of topological lemmas, where the construction of coproducts and free objects (over either sets or spaces) is carried out by the completion of certain algebras with respect to suitable totally bounded uniformities. On the other hand, they can also be established, albeit in a somewhat less explicit way, by categorical arguments involving the Adjoint Functor Theorem and the notion of tripleability (Manes [10]). For a further approach to coproducts, see also Golema [4].

Finally, we note about the category **KA** that its monomorphisms are exactly

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[2]) In general, this no longer holds for infinitary algebras.

[3]) The argument here employs compactness; whether the same is true for arbitrary A we do not know.

its one-to-one maps and hence the same as its embeddings, i.e. the maps which provide isomorphisms with the image, and dually, its coequalizers are exactly its onto maps.

The subcategory $\mathbf{K}_0\mathbf{A}$ of **KA** is evidently productive and hereditary, and hence epireflective, in **KA** (Herrlich [7]). Moreover, the reflection $\mathbf{KA} \to \mathbf{K}_0\mathbf{A}$ is provided in a very natural topological manner: One proves that, on any topological algebra, the connected component relation is a congruence, and hence one can take quotients in the present setting; passage from $A \in \mathbf{KA}$ to the resulting algebra of its connected components then provides the reflection to $\mathbf{K}_0\mathbf{A}$, the quotient maps giving the adjunction.

We now turn to the topological algebras we are specifically concerned with here. In the following, **A** will always be a fixed equational class, and **C** a subclass of **FA** which is hereditary and finitely productive, tacitly assumed to be non-trivial, i.e. to contain algebras with more than one element. With this, let $\mathfrak{R}A$, for any $A \in \mathbf{KA}$, be the set of those closed congruences Θ on A for which $A/\Theta \in \mathbf{C}$, and $\mathbf{ProC} \subseteq \mathbf{KA}$ the subclass of those A for which the intersection of all $\Theta \in \mathfrak{R}A$ is trivial, i.e. the diagonal Δ of $A \times A$. Note that, in a more categorical way, $A \in \mathbf{KA}$ belongs to **ProC** iff for any distinct maps $f, g: B \to A$ in **KA** there exists a map $h: A \to C$ for some $C \in \mathbf{C}$ such that $hf \neq hg$. The $A \in \mathbf{ProC}$ will be called the pro-**C** algebras.

Proposition 1. (1) **ProC** *is hereditary and productive in* **KA**.

(2) $\mathbf{ProC} \subseteq \mathbf{K}_0\mathbf{A}$.

(3) $A \in \mathbf{ProC}$ *iff* **A** *is a closed topological subalgebra of a product of algebras in* **C**.

(4) $A \in \mathbf{ProC}$ *iff* A *is a projective limit of algebras in* **C**.

This can be proved most directly from the definition of **ProC** by means of the $\mathfrak{R}A$, but it can also be obtained as a formal consequence of certain categorical properties of **KA**.

It follows from this proposition that **ProC** is epireflective in **KA**, the reflection being provided by passing from A to its quotient modulo the congruence $\bigcap\Theta(\Theta \in \mathfrak{R}A)$; moreover, **ProC** is actually the epireflective *hull* of **C** in **KA**.

Concerning $\mathfrak{R}\mathbf{A}$ one has, for any $A \in \mathbf{ProC}$: (1) Each $\Theta \in \mathfrak{R}A$ is open, and thus open-closed, in $A \times A$. (2) $\mathfrak{R}A$ is closed under finite intersections and hence a basis for the uniformity of A. (3) If **C** is also closed under quotients then $\mathfrak{R}A$ consists of all closed congruences of finite index on A, and these are exactly the congruences on A open-closed in $A \times A$.

Possibly the most typical examples of pro-**C** algebras are those where $\mathbf{C} = \mathbf{FA}$, such as the class **ProFG** of profinite groups, **ProFAb** of profinite abelian groups, or **ProFAnn** of profinite commutative rings with unit. However, one also has naturally occuring cases where $\mathbf{C} \subset \mathbf{FA}$, such as the class $p\mathbf{G}$ of (finite) p-groups leading to the pro-p groups.

Regarding **ProFA**, there are instances where this is topologically characterized in **KA** as $\mathbf{K}_0\mathbf{A}$; some examples are groups, semigroups, Boolean lattices, and associative rings. We have no general result concerning the identity $\mathbf{ProFA} = \mathbf{K}_0\mathbf{A}$ apart from the obvious remark that any equational subclass of an equational class with this property inherits it; on the other hand, there are the following counterexamples: $\beta\mathbf{N}$ together with the continuous extension of the successor function $k \rightsquigarrow k + 1$ is a zero-dimensional compact algebra, of type (1), but not profinite since it has only countably many open-closed congruences and $\beta\mathbf{N}$ is not metrizable. Similarly, for any finitely generated infinite abelian group G, the Stone-Čech compactification of its underlying set, made into a topological G-set by extending the left translations of G continuously, is a counterexample for the same reason. Another such example, with countably many unary operations on the one-point compactification of **N**, is given in Eastman [3].

2. Properties transferred from C to ProC. Here we collect a number of properties, primarily of a categorical nature, which are invariant under passage from **C** to **ProC**.

Proposition 2. *If all epimorphisms are onto in* **C** *then this also holds in* **ProC**.

Examples of classes **C** to which this applies are the **FA** for the equational classes **A** consisting of the following kinds of algebras: Boolean algebras; commutative rings with unit satisfying the equations $x^p = x$, $px = 0$ for some prime p; commutative rings with unit satisfying the equation $x^n = x$ for some n; groups; abelian groups. Non-equationally defined such **C** are given by the p-groups, the nilpotent groups, and the finite semi-prime commutative rings with unit. On the other hand, finite distributive lattices and finite semigroups have epimorphisms which are not onto.

That epimorphisms in **ProC** are onto whenever they are so in **C** is not a purely categorical consequence of the fact that **ProC** is the epireflective hull of **C** in some category: the class of discrete spaces and its epireflective hull in all Hausdorff spaces provide a counterexample.

In the equational class **A**, the monomorphisms are exactly the one-to-one maps, as one readily sees from the existence of free algebras in **A**. Furthermore, this carries over to **C** as well as to **ProC**. Equalizers are special monomorphisms, and it is noteworthy when these two coincide in a category.

Proposition 3. *If all monomorphisms are equalizers in* **C** *then this also holds in* **ProC**.

Here, finite groups, abelian groups, and M-sets for any monoid M provide examples of classes **C** with the stated property, and finite semigroups and distributive lattices counterexamples. Also, this proposition rests on more than the fact that **ProC** is an epireflective hull of **C**, as can again be seen by looking at discrete spaces in all Hausdorff spaces.

The most significant of the additional hypothesis on **A** which will be used here is described as follows: Two congruences Θ and Λ, on any algebra, are called *permutable* iff $\Theta \circ \Lambda = \Lambda \circ \Theta$, and if this is so then $\Theta \vee \Lambda = \Theta \circ \Lambda$, the join $\vee$ referring to the lattice of all congruences on the algebra. The equational class **A** will be called *congruence permutable* iff any two congruences on any algebra in **A** are permutable. The importance of this condition for universal topological algebra was first noted in Malcev [9]. We recall that congruence permutability is equivalent to the existence of a ternary polynomial p, i.e. an element in the absolutely free algebra, of the same arity type as **A**, with a three element basis $\{x, y, z\}$, such that the equations $p(x, x, z) = z$ and $p(x, z, z) = x$ hold in the class **A**, and that it implies modularity for the congruence lattice of each algebra in **A** (Malcev [8]). Natural examples of congruence permutable equational classes are given by groups, rings, modules, and Boolean lattices.

Proposition 4. *For congruence permutable* **A**, *if* **C** *is closed under quotients then* **ProC** *is also closed under quotients* (*in* **KA**).

For any congruence permutable **A**, **FA** has the required property and thus all such **ProFA** are closed under quotients. Other examples for such **C** are given by the p-groups, nilpotent groups, abelian p-groups, and the p-rings, i.e. the finite rings whose additive group is a p-group.[4]) That this proposition does not hold in general is shown by the example **A** = **Ens** ("algebras" without operations): Here **ProFA** is the class of zero-dimensional compact Hausdorff spaces and clearly not closed under quotients.

Next, for two subclasses **D** and **E** $\supseteq$ **D** of **A** (and, analogously, of **KA**), call **D** extensive in **E** iff any $B \in$ **E** already belongs to **D** whenever there exists a non-void $A \subseteq B$ and a congruence Θ on B such that $\Theta(A) = A$, and A and B/Θ belong to **D**.

Proposition 5. *For congruence permutable* **A** *and* **C** *closed under quotients, if* **C** *is extensive in* **FA** *then* **ProC** *is extensive in* **ProFA**.

Examples for this situation are provided by the p-groups, abelian p-groups, and p-rings.

With additional hypotheses on **A** one can reach the stronger conclusion that **ProC** is extensive in **KA**: this is the case whenever **A** is congruence regular, i.e. the algebras in **A** have the property that the only congruence with a singleton class is the trivial one, and **ProFA** = $\mathbf{K}_0$**A**. The above examples, of course, all come under this.

Given a category **K**, we shall mean by a (*reflective*) *factorization* of **K** a family $(\mathbf{K}_i)_{i \in J}$ of reflective subcategories $\mathbf{K}_i$ of **K** such that the adjunctions $\varrho_i: A \to R_i A$, R_i the functor reflecting **K** into $\mathbf{K}_i$, provide an isomorphism $\varrho : A \to \prod R_i A$ for each

[4]) This term is sometimes used with a different meaning, and these rings are also called the finite p-torsion rings.

$A \in \mathbf{K}$. Further, we shall call such a factorization *disjoint* iff the only maps $A \to B$ where $A \in \mathbf{K}_i$ and $B \in \mathbf{K}_j$ for $i \neq j$ are those that factor through the terminal object of $\mathbf{K}$. It is clear that such factorizations reduce the structure of $\mathbf{K}$ to a good extent to that of the product category $\prod \mathbf{K}_i$.

In the following, let $(\mathbf{C}_i)$ be a family of subclasses of $\mathbf{C}$, each hereditary and finitely productive and hence reflective with onto maps as adjunctions; then the latter also holds for the subcategories $\mathbf{ProC}_i$ of $\mathbf{ProC}$, and one has:

Proposition 6. *If* $(\mathbf{C}_i)$ *is a disjoint factorization of* $\mathbf{C}$ *then* $(\mathbf{ProC}_i)$ *is a disjoint factorization of* $\mathbf{ProC}$.

Examples of classes $\mathbf{C}$ with disjoint factorizations are given by the finite abelian groups, the nilpotent groups, and the finite rings: for the first and second, the subclasses consist of the p-groups, in the original class, for each prime p, and for the third of the p-rings. A ready way of proving disjointness for a factorization $(\mathbf{C}_i)$ of $\mathbf{C}$ is to show that the $\mathbf{C}_i$ are closed under quotients and $\mathbf{C}_i \cap \mathbf{C}_j$, for $i \neq j$, contains only trivial (i.e. singleton) algebras.

We call an algebra $A \in \mathbf{C}$ *semi-simple* iff the intersection of its maximal (proper) congruences is trivial, and use the term analogously for $A \in \mathbf{ProC}$, the congruences to be admitted being the open ones. Of course, a maximal open congruence on an $A \in \mathbf{ProC}$ is, in fact, a maximal congruence.

Proposition 7. *If all* $C \in \mathbf{C}$ *are semi-simple then so are all* $A \in \mathbf{ProC}$.

Examples of classes $\mathbf{C}$ all whose members are semi-simple are evidently easy to obtain: In general, whenever $\mathbf{FA}$ contains any non-trivial algebras, it also contains non-trivial semi-simple ones, and they form a hereditary and finitely productive class. Particular instances are given by all finite abelian groups of squarefree index, all finite distributive lattices, and all finite commutative rings with unit satisfying the equation $x^n = x$ for some n.

3. Completions. This section deals with the construction of pro-$\mathbf{C}$ algebras from algebras in $\mathbf{A}$ by means of uniform space completion and a number of applications of this process.

We shall call an algebra $A \in \mathbf{A}$ $\mathbf{C}$-separated iff its congruences Θ for which $A/\Theta \in \mathbf{C}$ have trivial intersection. The class $\mathbf{TC}$ of all $\mathbf{C}$-separated $A \in \mathbf{A}$ is then hereditary and productive in $\mathbf{A}$ and consists of all algebras isomorphic to subalgebras of products of algebras in $\mathbf{C}$. Consequently, $\mathbf{TC}$ is reflective in $\mathbf{A}$, the reflection R: $\mathbf{A} \to \mathbf{TC}$ again given by passage to quotients, i.e. $RA = A/\Lambda$ where Λ is the intersection of all congruences Θ on A such that $A/\Theta \in \mathbf{C}$, and the adjunction by the corresponding quotient maps. Note, incidentally, that the underlying algebra of any $A \in \mathbf{ProC}$ belongs to $\mathbf{TC}$.

Now consider, on a given $A \in \mathbf{TC}$, any filter basis $\mathfrak{F}$ of congruences Θ such that $A/\Theta \in \mathbf{C}$ which has trivial intersection. Then $\mathfrak{F}$ is a basis of a separated uniformity on the underlying set of A, and all operations of A are uniformly continuous with respect to this since congruences are subalgebras of $A \times A$. This uniformity is, moreover, totally bounded, and hence the completion of the resulting uniform space is compact. The operations of A, by their uniform continuity, then have continuous extensions to the completed space, and whatever equations they satisfy in A the latter will also satisfy, by the familiar principle of extension of identities. It follows that one obtains a topological algebra $\tilde{A} \in \mathbf{KA}$ in this manner.

For more information about these $\tilde{A}$, note that the quotient map $\nu: A \to A/\Theta$, where $\Theta \in \mathfrak{F}$, has a continuous extension $\tilde{\nu}: \tilde{A} \to A/\Theta$ by uniform continuity (discrete uniformity on A/Θ), and therefore $\tilde{A}/\tilde{\Theta} \cong A/\Theta$ for the kernel congruence $\tilde{\Theta}$ of $\tilde{\nu}$. Since $\tilde{\Theta}$ is also the closure of Θ in $\tilde{A}$, $\bigcap\tilde{\Theta}(\Theta \in \mathfrak{F})$ is trivial. It follows that all $\tilde{\Theta}$ for $\Theta \in \mathfrak{F}$ belong to $\mathfrak{N}\tilde{A}$ and $\tilde{A} \in \mathbf{ProC}$. We call $\tilde{A}$ the *$\mathfrak{F}$-completion* of A. For the set of *all* Θ with $A/\Theta \in \mathbf{C}$ this completion will be called the *$\mathbf{C}$-adic completion* of A, and the uniformity involved the *$\mathbf{C}$-adic uniformity* of A.

Note that $\tilde{A} = \varprojlim_{\Theta\in\mathfrak{F}} \tilde{A}/\tilde{\Theta} = \varprojlim_{\Theta\in\mathfrak{F}} A/\Theta$, the inverse systems involved being the obvious ones, so that $\tilde{A}$, for given $\mathfrak{F}$, could also be described without reference to uniformities and completions; there does, however, seem to be some technical advantage in making use of the latter.

For any $A \in \mathbf{TC}$, let PA be its $\mathbf{C}$-adic completion. Then, any $h: A \to B$ in $\mathbf{TC}$, being evidently uniformly continuous with respect to the $\mathbf{C}$-adic uniformities, extends uniquely to a homomorphism $Ph: PA \to PB$ of topological algebras, and the correspondence $A \rightsquigarrow PA$ and $h \rightsquigarrow Ph$ clearly provides a functor $P: \mathbf{TC} \to \mathbf{ProC}$.

Proposition 8. *The functor P of $\mathbf{C}$-adic completion is left adjoint to the underlying algebra functor* $\mathbf{ProC} \to \mathbf{TC}$.

Recall that the inclusion functor $\mathbf{TC} \to \mathbf{A}$ has the reflection $R: \mathbf{A} \to \mathbf{TC}$ as left adjoint. If one defines the generalized $\mathbf{C}$-adic completion by the functor $PR: \mathbf{A} \to \mathbf{ProC}$ one has, for purely formal reasons:

Corollary 1. *The underlying algebra functor* $\mathbf{ProC} \to \mathbf{A}$ *has the functor PR of generalized $\mathbf{C}$-adic completion as left adjoint.*

In similar manner it follows that the underlying set functor $\mathbf{ProC} \to \mathbf{Ens}$ has PRF as left adjoint, $F: \mathbf{Ens} \to \mathbf{A}$ being the free algebra functor. Note that there are pro-$\mathbf{C}$ algebras of arbitrarily large cardinality, so that the front adjunction is one-to-one.

Corollary 2. *For any set X there exists a free pro-$\mathbf{C}$ algebra with basis X, provided by the generalized $\mathbf{C}$-adic completion of the free algebra in $\mathbf{A}$ with basis X.*

In some cases, the free algebras in $\mathbf{A}$ are known to belong to $\mathbf{TC}$, which somewhat simplifies the description of the free pro-$\mathbf{C}$ algebras. For instance, this holds for both, $\mathbf{FG}$ and $p\mathbf{G}$ in $\mathbf{G}$, for $\mathbf{FAb}$ and $p\mathbf{Ab}$ in $\mathbf{Ab}$, and for $\mathbf{FAnn}$ and $p\mathbf{Ann}$ in $\mathbf{Ann}$.

Let $(\mathbf{C}_i)$ now be a factorization of $\mathbf{C}$, R_i: $\mathbf{A} \to \mathbf{TC}_i$ the reflections, P_i: $\mathbf{TC}_i \to \mathbf{ProC}_i$ the $\mathbf{C}_i$-adic completion functor. Then one has:

Corollary 3. *For any $A \in \mathbf{A}$, PRA is isomorphic to $\prod P_i R_i A$, natural in A.*

Similarly, if E and E_i are the free pro-$\mathbf{C}$, and free pro-$\mathbf{C}_i$ algebra functors on **Ens**:

Corollary 4. *For any set X, EX is isomorphic to $\prod E_i X$, natural in X.*

Thus, one has that the free profinite abelián group on X is the product of the free pro-p abelian groups on X, and the free pro-nilpotent group on X is the product of the free pro-p groups on X, and similar consequences.

Consider now any zero-dimensional Hausdorff space X. If F_X is the $\mathbf{TC}$-reflection of the algebra in $\mathbf{A}$ with the underlying set of X as basis (which may be assumed to contain that set), let $\mathfrak{F}_X$ be the filter basis of those congruences Θ on F_X for which $F_X/\Theta \in \mathbf{C}$, and which induce on X an open-closed decomposition. One proves that $\bigcap \Theta(\Theta \in \mathfrak{F}_X)$ is trivial and hence has the $\mathfrak{F}_X$-completion GX of F_X. Futhermore, any continuous map u: $X \to Y$ between zero-dimensional Hausdorff spaces induces a homomorphism $\bar{u}$: $F_X \to F_Y$ which is uniformly continuous for the uniformities given by $\mathfrak{F}_X$ and $\mathfrak{F}_Y$, respectively, and hence extends to a homomorphism Gu: $GX \to GY$ in **ProC**. The result is a functor G into **ProC**.

Proposition 9. *The functor G is left adjoint to the underlying space functor from* **ProC**, *and the front adjunction provides embeddings.*

Of course, for a factorization $(\mathbf{C}_i)$ of $\mathbf{C}$ one again has $GX \cong \prod G_i X$, analogous to the case of free pro-$\mathbf{C}$ algebras over sets.

Since any zero-dimensional Hausdorff space X can be regarded as a subspace of the pro-$\mathbf{C}$ algebra GX one may ask what its closure is in GX, this providing a particular compactification procedure for all zero-dimensional Hausdorff spaces. The answer is simple: It is just the maximal zero-dimensional compactification of X.

Corollary. *If A is a free pro-*$\mathbf{C}$ *algebra with basis space X and B the closed subalgebra of A generated by a subspace Y of X then B is a free pro-*$\mathbf{C}$ *algebra on Y iff every finite open-closed partition of Y is the restriction of such a partition of X.*

Completions can also be used to obtain coproducts: Given any family (A_λ) of pro-$\mathbf{C}$ algebras, let F be the coproduct in $\mathbf{A}$ of their underlying algebras, with algebra homomorphism u_λ: $A_\lambda \to F$ as the canonical maps. Then consider the filter basis $\mathfrak{F}$ of congruences Θ on F for which $F/\Theta \in \mathbf{C}$ and $(u_\lambda \times u_\lambda)^{-1}(\Theta) \in \mathfrak{N} A_\lambda$ for all λ. For $\Lambda = \bigcap \Theta(\Theta \in \mathfrak{F})$, let $F_0 = F/\Lambda$, $\mathfrak{F}_0$ the filter basis of congruences Θ_0 induced modulo Λ on F_0 by the $\Theta \in \mathfrak{F}$, A the $\mathfrak{F}_0$-completion of F_0, and j_λ: $A_\lambda \to A$ the map resulting from u_λ.

Proposition 10. *A is the coproduct of (A_λ) in* **ProC** *with (j_λ) as its family of canonical maps.*

Note that the existence of coproducts in **ProC** and of the adjoint functors considered above could also be obtained from general principles (reflection from **KA**, Adjoint Functor Theorem), but this would yield rather less information than the constructions by means of completion provide.

We conclude this section with a result concerning the structure of certain individual completions. Let Σ be a set of filter bases $\mathfrak{F}$ of congruences on some algebra $A \in \mathbf{TC}$ such that $A/\Theta \in \mathbf{C}$ for each $\Theta \in \mathfrak{F}$ and $\bigcap \Theta(\Theta \in \mathfrak{F}) = \Delta$. Σ will be called independent iff $\Theta_0 \circ (\Theta_1 \cap \ldots \cap \Theta_n) = \nabla$, the congruence with a single class, for any $\Theta_i \in \mathfrak{F}_i \in \Sigma$ where all $\mathfrak{F}_i$ are distinct.

Proposition 11. *For independent Σ, the product of the $\mathfrak{F}$-completions of A, for $\mathfrak{F} \in \Sigma$, is isomorphic to the $\mathfrak{G}$-completion of A, $\mathfrak{G}$ consisting of all $\Theta_1 \cap \ldots \cap \Theta_n$ where $\Theta_i \in \mathfrak{F}_i \in \Sigma$, the isomorphism being the extension, to the $\mathfrak{G}$-completion, of the diagonal map of A into the product.*

4. Projectives. In the present context, projectivity is always understood with respect to onto maps. Since all topological algebras under consideration are compact, this makes the general theory given in Banaschewski [1] applicable, and one has the following:

(1) *$A \in$ **ProC** is projective iff every onto map $B \to A$ in **ProC** has a right inverse iff every coessential onto map $B \to A$ in **ProC** is an isomorphism, a map being coessential onto iff it is onto and maps proper closed subalgebras to proper subalgebras.*[5])

(2) *For any $A \in$ **ProC** there exists an essentially unique coessential onto map h: $B \to A$ with projective B. B, or sometimes h, is called a projective cover of A.*

(3) *For any onto map h: $B \to A$ in **ProC**: B is a projective cover of A iff h is coessential onto and any map g in **ProC** for which hg is coessential onto is an isomorphism iff any onto map g in **ProC** with projective codomain and a factorization $h = fg$ is an isomorphism.*

Our aim here is to obtain statements about the projectives in **ProC** on the basis of algebraic conditions on **A** and **C**. We make one blanket hypothesis: **A** is taken to be congruence permutable, **C** closed under quotients, and for the congruences in **A** it is further assumed that for any algebras A, B and C in **A** such that $B \subseteq C \subseteq A$, and any congruence Θ on A, $\Theta(B) = B$ implies $\Theta(C) = C$. Note that these conditions, as far as they concern **A**, are satisfied by any equational class each of whose algebras

[5]) In Banaschewski [1], "essential" was used in place of "coessential"; the latter seems preferable in the context of projectivity.

has a group as a reduct, typical instances being groups themselves, modules over a ring, rings, and algebras over a ring (associative or not).

A preliminary result, which only depends on the first two parts of the present assumption, is:

Proposition 12. *$A \in$ **ProC** is semi-simple iff it is a product of simple $C \in$ **C**.*

The proof uses the fact that congruence permutability implies the modularity of the congruence lattices for all algebras in **A**.

In dealing with projectivity it is desirable to have a characterization of the coessential maps. To this end, one first proves that (i) any closed proper subalgebra of an $A \in$ **ProC** is contained in a maximal closed subalgebra, and (ii) for any maximal closed subalgebra M of A there exist congruences $\Theta \in \Re A$ such that $\Theta(M) = M$ and among these a largest one, say Θ_M. The intersection of all Θ_M will then be called the *Frattini congruence* ΦA of A. Its significance is as follows:

Lemma. *In* **ProC**, *an onto map $h: A \to B$ is coessential iff* $\operatorname{Ker}(h) \subseteq \Phi A$.

Corollary 1. *For projective A, $B \in$ **ProC**, $A \cong B$ iff $A/\Phi A \cong B/\Phi B$.*

Corollary 2. *A, $B \in$ **ProC** have isomorphic projective covers iff $A/\Phi A \cong B/\Phi B$.*

Note also: $\Phi(A/\Phi A)$ is trivial for any $A \in$ **ProC**.

Applications of the Frattini congruence, in the case of profinite groups, occur in Banaschewski [1] and in Gruenberg [6].

Proposition 13. *If all $C \in$ **C** are semi-simple, and each simple $S \in$ **C** has a proper subalgebra then all $A \in$ **ProC** are projective.*

The proof of this and the following proposition proceeds by showing that the hypotheses imply $\Phi A = \Delta$ for all A, which eliminates all non-trivial coessential maps.

Instances of classes **C** to which this is applicable are given by the finite abelian groups of square free index and analogous classes of finite modules over a given ring.

Any algebra $C = A/\Theta_M$ is a finite algebra containing a maximal subalgebra D such that the only congruence Θ on C for which $\Theta(D) = D$ is the trivial one. We call such algebras *compressed.* Note that any simple algebra which has proper subalgebras is compressed, but one can easily give example of compressed algebras which are not simple. In the other direction, let an algebra S be called *strongly simple* iff the diagonal $\Delta \subseteq S \times S$ is a maximal subalgebra (rather than merely a maximal congruence, which means simplicity). One then has, for finite S, $S \times S$ is compressed whenever $S \in$ **A** is strongly simple and **A** congruence regular.

Proposition 14. *If* **A** *is congruence regular and all $C \in$ **C** are powers of a single strongly simple $S \in$ **C** then all non-trivial $A \in$ **ProC** are projective.*

A particular instance of this, which does not fall under the previous proposition, is the fact that all non-trivial profinite commutative rings satisfying the equations $x^p = x$, $px = 0$ for some prime p are projective (in their category) which was proved by an entirely different method in Banaschewski [1].[6])

We now turn to the relationship between projective and free pro-**C** algebras. Clearly, any pro-**C** algebra free on a set is projective (the maps admitted are onto), whereas, conversely, the last two propositions readily lead to cases in which not every projective is free on a set, or, for that matter, on a space. Rather more surprising is:

Proposition 15. *Any pro-***C** *algebra free on some zero-dimensional Hausdorff space is projective.*

The proof of this depends strongly on the assumption of congruence permutability and is related to the corresponding proof for profinite groups in Serre [12].

Since a projective pro-**C** algebra A is determined by $A/\Phi A$ one expects information on projectives if one has some on the pro-**C** algebras of the form $A/\Phi A$. Take, for instance, **C** to have, up to isomorphism, only one compressed algrebra S, and assume further that S is simple. This makes each $A/\Phi A$ isomorphic to some power $S^{\mathfrak{m}}$, $\mathfrak{m}$ uniquely determined by A. We call $\mathfrak{m}$ the *colength* of A and note that a projective pro-**C** algebra is then simply determined by its colength. Considering the colengths of free pro-**C** algebras one obtains:

Proposition 16. *If all compressed algebras in* **C** *are simple and isomorphic to each other then the projective pro-***C** *algebras with infinite colength* $\mathfrak{m}$ *are exactly the pro-***C** *algebras free on the one-point compactification of a discrete space of cardinality* $\mathfrak{m}$.

It should be added that not all projectives of infinite colength are free on some set; the latter are exactly those of colength $2^{\mathfrak{m}}$.

What the situation is regarding the projectives of finite colength we do not know. A priori it would seem possible that only certain natural numbers occur as the colengths of free pro-**C** algebras, but this is not the case for pro-p groups, abelian pro-p groups, and the categories of profinite modules analogous to the latter, to which the above applies: there, the colength of the free pro-**C** algebra on a finite set of n elements is n, and the equation free = projective holds completely (Banaschewski [1]).

We conclude with a result concerning factorizations and projectives.

Proposition 17. *If* (**ProC**$_i$) *is a factorization of* **ProC**, *resulting from a factorization* (**C**$_i$) *of* **C**, *with reflections* R_i: **ProC** $\to$ **ProC**$_i$, *then* $A \in$ **ProC** *is projective iff all* R_iA *are projective in* **ProC**$_i$.

[6]) Erratum: The qualification "non-trivial" was omitted there, but the trivial ring is evidently not projective here. In general, the status of the trivial $A \in$ **ProC** depends on **C**: for any **C** $\subseteq$ **G**, for instance, the trivial groups are projective in **ProC** since they are initial besides being terminal objects.

As an immediate consequence of this and Proposition 14 one has, for instance, that all non-trivial profinite rings, for any equational class of rings generated by a set of finite prime fields, are projective. Similarly, but with some additional arguments, one obtains that projective pronilpotent groups are exactly the products of projective pro-p groups.

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