Kazimierz Kuratowski
A general approach to the theory of set-valued mappings


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A GENERAL APPROACH TO THE THEORY
OF SET-VALUED MAPPINGS

K. KURATOWSKI

Warszawa

Summary. Given an arbitrary set $Y$, a countably additive lattice $L$ of subsets of $Y$, and a metric separable space $X$, we consider set-valued mappings $F : Y \rightarrow \mathcal{C}(X)$ (where $\mathcal{C}(X)$ is the space of all compact subsets of $X$) satisfying either condition (3) or (6) below. This is a far going generalization of upper resp. lower semi-continuous mappings (case where $L$ is the lattice of all open subsets of $Y$). Other important applications are obtained by substituting to $L$ the lattice of all Borel sets of additive class $\alpha$, the lattice of measurable sets, of projective sets etc.

I. Introduction

1. Definitions. Let $Y$ be a set of arbitrary elements and $L$ a countably additive lattice (i.e. closed under countable unions) of subsets of $Y$ containing as members the empty set and the set $Y$.

Denote $(-L) = \{E : (Y - E) \in L\}$.

Denote by $A_\sigma$ (resp. $A_\delta$) the lattice generated by the family of sets $A$ and closed under countable unions (resp. intersections).

Let $Z$ be a topological space. A mapping $f : Y \rightarrow Z$ will be called an $L$-mapping, briefly $f \in L^0$, if

(1) $f^{-1}(G) \in L$ whenever $G$ is open in $Z$ ;

equivalently: if

(2) $f^{-1}(K) \in (-L)$ whenever $K$ is closed in $Z$

(compare [4], Chapter IX).

2. Examples. In the following examples $Y$ is assumed to be a topological space.

1. Let $L$ be the lattice $G$ of all open subsets of $Y$. Then the mapping $f : Y \rightarrow Z$ is an $L$-mapping iff $f$ is continuous.

2. Let $L$ be the lattice of all Borel subsets of $Y$. Then $f$ is an $L$-mapping iff it is $B$-measurable (a Baire mapping).
3. Let $L$ be the lattice of Borel subsets of $Y$ of additive class $\alpha < \Omega$ (recall that the families: of open sets, of $F_\sigma$-sets, of $G_\delta$-sets etc. are additive Borel classes; the families of closed sets, of $G_\sigma$-sets, of $F_\delta$-sets etc. are multiplicative Borel classes). Here an $L$-mapping means $B$-measurable (Baire mapping) of class $\alpha$. (For an outline of a theory of set-valued $B$-measurable mappings, see [9].)

4. Let $Y$ be a Polish space (= complete separable) and let $L$ denote its $n$th projective class; recall that the 0-projective class is the class of all Borel sets, the first projective class (the class of Souslin sets or $\mathcal{A}$-sets) is composed of continuous images of sets of class 0, the second projective class (the class of $\mathcal{C}_0$-sets) is composed of the complements to the sets of the first projective class and so on; in general, the projective class $2n + 1$ consists of continuous images of sets of class $2n$ and the $2n$-class consists of complements of sets of class $2n - 1$.

Note that the projective classes are closed under countable union and countable intersection. If $Z$ has a countable open base, then, if $f : Y \to Z$ is an $L$-mapping, i.e. if $f^{-1}(G)$ is of projective class $n$ whenever $G$ is open, then $f^{-1}(G)$ is also of class $n + 1$.

In particular, if $n = 1$, $f$ is $B$-measurable (because by a known theorem of Souslin, a set which is simultaneously $\mathcal{A}$ and $\mathcal{C}_0$ is a Borel set).

5. Let $L$ be the lattice of measurable sets (more generally: a $\sigma$-algebra of subsets of $Y$). Then an $L$-mapping means $L$-measurable mapping.

II. Compact-valued mappings

We are going to consider set-valued mappings (called also multifunctions) $F : Y \to \mathcal{C}(X)$, where $Y$ is an arbitrary set, $X$ a normal space with a countable open base (equivalently: a metric separable space) and $\mathcal{C}(X)$ the space of all compact subsets of $X$. Thus $F : Y \to \mathcal{C}(X)$ means that for each $y \in Y$, $F(y)$ is a compact subset of $X$ (of course, if $X$ is compact, then $\mathcal{C}(X) = 2^X$, space of all closed subsets of $X$, and $F(y)$ is a closed subset of $X$).

The space $\mathcal{C}(X)$ is endowed with the Vietoris topology, which means (compare [11]) that the collection of all sets which are either of the form

(i) \{ $F : F \subset G$ \} or
(ii) \{ $F : F \cap G \neq \emptyset$ \},

where $F$ is compact and $G$ open in $X$, is an open subbase of $\mathcal{C}(X)$. Therefore, the collection of all sets of the form

(iii) \{ $F : F \subset G$ \} $\cap$ \{ $F : F \subset G_1 \neq \emptyset$ \} $\cap$ \ldots $\cap$ \{ $F : F \subset G_n \neq \emptyset$ \}

is an open base of $\mathcal{C}(X)$. Finally, since $X$ has a countable base, so does $\mathcal{C}(X)$, and hence

(iv) every open subset of $\mathcal{C}(X)$ is a countable union of sets of the form (iii).
3. Definition. Given a lattice $L$ (like in § 1), the mapping $F : Y \to \mathcal{G}(X)$ will be called of class $L^+$ or briefly, $F \in L^+$, if $F^{-1}(\mathcal{G}(G)) \in L$, i.e., if

$$\{y : F(y) \subset G\} \in L \text{ whenever } G \text{ is open in } X,$$

equivalently, if

$$\{y : F(y) \cap K = \emptyset\} \in L \text{ whenever } K \text{ is closed in } X.$$

Symmetrically, $F \in L_-$, if $F^{-1}(\mathcal{G}(K)) \in (-L)$, i.e., if

$$\{y : F(y) \subset K\} \in (-L) \text{ whenever } K \text{ is closed in } X,$$

equivalently, if

$$\{y : F(y) \cap G = \emptyset\} \in (-L) \text{ whenever } G \text{ is open in } X.$$

Let us note that, according to (1), $F \in L^0$ iff $F^{-1}(G) \in L$ for each $G$ open in $2^X$.

4. Examples and Remarks. 1. Let $Y$ be a topological space, $X$ compact and $L$ the lattice of all open subsets of $Y$. Then $F \in L^+$ means that $F$ is upper semi-continuous and $F \in L_-$ means that $F$ is lower semi-continuous.

2. Let $Y$ be a metric space, $X$ compact and $L$ the additive Borel class $\mathcal{B}(X)$. Then $F \in L^+$ means that the sets $F^{-1}(2^G)$ are of additive class $\alpha$. Similarly $F \in L_-$ means that the sets $F^{-1}(2^K)$ are of multiplicative class $\alpha$.

(Instead of $F \in L^+$ ($F \in L_-$) we also say that $F$ is of Baire class $\alpha^+$ (class $\alpha^-$).)

5. Elementary properties of classes $L^+$ and $L_-$.

Theorem 1. If $F$ is constant, say $F(y) = K_0$ for each $y \in Y$, then $F \in L^0$.

Because $F^{-1}(G) = Y$ or $\emptyset$ according to whether $K_0 \in G$ or $K_0 \notin G$. In both cases $F^{-1}(G) \in L$.

Theorem 2. Let $f : Y \to X$ and $F(y) = \{f(y)\}$ (i.e., $f$ is point-valued). If $F \in L^+ \cup L_-$, then $f$ is an $L$-mapping.

This follows easily from the formula

$$\{y : F(y) \subset A\} = \{y : f(y) \in A\} = f^{-1}(A)$$

and from (3) and (1), resp. from (5) and (2) (substituting $A = G$ or $A = K$, resp.).

Theorem 3. $L^0 \subset L^+ \cap L_-$. (Here $L$ is not assumed to be countably additive.)

This follows from (1) and (3), resp. from (2) and (4), because $\mathcal{G}(G)$ is open, and $\mathcal{G}(K)$ is closed in $\mathcal{G}(X)$. 
Theorem 4. $L^+ \cap L_- \subseteq L^0$. Hence $L^0 = L^+ \cap L_-$.

Proof. Let $G$ be an open subset of $\mathcal{O}(X)$. Let $F \in L^+ \cap L_-$. We have to show that $F^{-1}(G) \in L$. Now by (iv), $F^{-1}(G)$ is a countable union of sets of the form

$$(7) \quad \{y : F(y) \subseteq G\} \cap \{y : F(y) \cap G_1 \neq \emptyset\} \cap \ldots \cap \{y : F(y) \cap G_n \neq \emptyset\}$$

and by (3) and (6) each of the factors of (7) is a member of $L$, and therefore the set (7) belongs to $L$. Hence $F^{-1}(G) \in L$.

Theorem 5. $L^+ \subseteq ((-L)_a)^- \text{ and } L_- \subseteq ((-L)_a)^+$.

Proof. 1. Let $F \in L^+$ and let $K$ be closed in $X$. We have to show that $\{y : F(y) \subseteq K\} \in (-L)_a$, i.e. that $\{y : F(y) \subseteq K\} \in L_a$. Put $K = G_1 \cap G_2 \cap \ldots$ where $G_n$ is open. Then

$$(8) \quad \{y : F(y) \subseteq K\} = \bigcap_n \{y : F(y) \subseteq G_n\}$$

and the proof is completed because $\{y : F(y) \subseteq G_n\} \in L$.

2. Let $F \in L_-$ and let $G$ be open in $X$. We have to show that $\{y : F(y) \subseteq G\} \in (-L)_a$. Put $G = K_1 \cup K_2 \cup \ldots$ where $K_n$ is closed and $K_n \subset \text{Int}(K_{n+1})$. If $F(y) \subseteq G$, then — by compactness of $F(y)$ — there is $n$ such that $F(y) \subseteq K_n$. Thus

$$(8') \quad \{y : F(y) \subseteq G\} = \bigcup_n \{y : F(y) \subseteq K_n\}$$

and the proof is completed because $\{y : F(y) \subseteq K_n\} \in (-L)$.

Corollary 5'. If $L = -L$ (i.e. if $L$ is a $\sigma$-algebra), then $L^+ = L_-$.

Corollary 5". If $L \subseteq (-L)_a$, then $(L^+ \cup L_-) \subseteq ((-L)_a)^0$.

This follows from Theorems 4 and 5 and the formula

$$L^+ \subseteq ((-L)_a)^+ \text{ and } L_- \subseteq ((-L)_a)^-$$

which is an obvious consequence of $L \subseteq (-L)_a$.

Remark. The assumption $L \subseteq (-L)_a$ is satisfied if, for example, $L$ denotes the $\alpha$ additive Borel class. So, in particular, it follows from Corollary 5" that semi-continuous compact-valued mappings are of the first Baire class.


Theorem 1. $L^+$ and $L_-$ are closed under the operation of finite union.

In other terms, if $F_j \in L^+$ for $j = 0, 1$, and $F = F_0 \cup F_1$, then $F \in L^+$. Similarly, if $F_j \in L_-$, then $F \in L_-$. 

Here $F = F_0 \cup F_1$ means that $F(y) = F_0(y) \cup F_1(y)$ for each $y \in Y$. (A similar meaning has $F = F_0 \cap F_1$.) The theorem follows immediately from the formula (compare [8], p. 20(2)):

$$F^{-1}(\mathcal{C}(A)) = F_0^{-1}(\mathcal{C}(A)) \cap F_1^{-1}(\mathcal{C}(A)),$$

where $A \subset X$ is open, respectively closed.

**Theorem 2.** $L^+$ is closed under countable intersections.

In other terms, if $F_n \in L^+$ for $n = 1, 2, \ldots$, then $(\bigcap F_n) \in L^+$.

The proof, completely similar to that of Theorem 8 of [9], is based on the following two general valid formulas (compare [8], p. 179(2)). Let $F = F_0 \cap F_1$.

Then

$$F^{-1}(\mathcal{C}(G)) = \bigcup_{i,j} [F_0^{-1}(\mathcal{C}(G \cup R_i)) \cap F_1^{-1}(\mathcal{C}(G \cup R_j))],$$

where $R_1 \cap R_j = \emptyset$, $R_1, R_2, \ldots$ being an open base of $X$ closed under finite unions, and

$$F^{-1}(\mathcal{C}(G)) = \bigcup_{n} F_n^{-1}(\mathcal{C}(G)),$$

where $F = F_0 \cap F_1 \cap \ldots$ and $F_0 \supset F_1 \supset \ldots$.

**Theorem 3.** $L_-$ is closed under the operation of the closure of a countable union.

More precisely, if $F = F_1 \cup F_2 \cup \ldots$ and $F_n \in L_-$ for $n = 1, 2, \ldots$, then $F \in L_-$, provided $F(y)$ is compact for each $y \in Y$.

This follows from the formula (compare [8], p. 164 (iii)):

$$F^{-1}(\mathcal{C}(A)) = \bigcap_{n} F_n^{-1}(\mathcal{C}(A)).$$

**Theorem 4.** If $F_0 \in L_-$ and $F_1 \in L^+$, then $\overline{F_0 - F_1} \in L_-.$

In particular, $\overline{X - F_1} \in L_-.$

This follows from the lemma which we are now going to prove (compare also [8], p. 181(2)):

**Lemma.** Let $F_0 : Y \to 2^X$, $F_1 : Y \to 2^X$ and $F = F_0 - F_1$. Let $K$ be closed in $X$ and let $R_1, R_2, \ldots$ be the sequence of members of an open base of $X$ such that $K \cap R_i = \emptyset$. Then

$$(9) \quad F^{-1}(2^X) = \bigcap_i \{Y - [F_1^{-1}(2^X - R_i) - F_0^{-1}(2^X - R_i)]\},$$

equivalently

$$(9') \quad Y - F^{-1}(2^X) = \bigcup_i \{F_1^{-1}(2^X - R_i) - F_0^{-1}(2^X - R_i)\}.$$
or

\[(9^*) \quad [F_0(y) - F_1(y) \notin K] \equiv \exists i : [F_1(y) \cap R_i = \emptyset \text{ and } F_0(y) \cap R_i \neq \emptyset].\]

**Proof.** 1. Let \(F_0(y) - F_1(y) \notin K\), i.e., \(F_0(y) \notin K \cup F_1(y)\). So let \(x \in F_0(y)\) and \(x \notin K \cup F_1(y)\). By the regularity of \(X\), there exists a member of the base, we may call it \(R_i\), such that

\[x \in R_i \quad \text{and} \quad R_i \cap (K \cup F_1(y)) = \emptyset.\]

It follows that \(F_1(y) \cap R_i = \emptyset\) and \(F_0(y) \cap R_i \neq \emptyset\), because \(x \in F_0(y) \cap R_i\).

Thus \(y\) satisfies the right member of \((9^*)\).

2. On the other hand, if \(y\) satisfies the right member of \((9^*)\) and \(K \cap R_i = \emptyset\), then \((F_0(y) - F_1(y)) \cap R_i = \emptyset\), because \((F_0(y) - F_1(y)) \cap R_i = (F_0(y) \cap R_i) - (F_1(y) \cap R_i) \neq \emptyset\).

Since \(R_i \subset X - K\), it follows that \((F_0(y) - F_1(y)) \cap (X - K) \neq \emptyset\), which means that the left member of \((9^*)\) is fulfilled.

**Remark.** As seen, we don’t require in our Lemma that the values of the mappings \(F_0\) and \(F_1\) be compact; they are only assumed to be closed in the (metric) space \(X\).

The same remark applies, of course, to Theorem 4.

**Theorem 5.** \(L^+\) and \(L_-\) are closed under the operation of limit

\[(10) \quad F = \lim F_n,\]

the convergence being uniform.

More precisely: if \(F_n \in L^+\) (resp. \(F_n \in L_-\)) for \(n = 1, 2, \ldots\), then \(F \in L^+\) (resp. \(F \in L_-\)), provided \(F(y)\) is compact for each \(y \in Y\).

Theorem 5 is a direct consequence of the following lemma which we are going to prove.

**Lemma.** Assume that the mappings \(F_n : Y \rightarrow \mathcal{C}(X)\), where \(n = 1, 2, \ldots\) and where \(X\) is metric, satisfy the condition \((10)\); in other terms, there is a sequence \(m_1 < < m_2 < \ldots\) such that

\[(11) \quad \text{dist} \ (F(y), F_j(y)) < 1/n \text{ for each } y \in Y \text{ and } j > m_n.\]

Then, we have, for each open \(G\) and closed \(K\),

\[\{y : F(y) \subset G\} = \bigcup_{n \ j > m_n} \bigcup_{y : F_j(y) \subset G_n}\]

\[\{y : F(y) \subset K\} = \bigcap_{n \ j > m_n} \bigcap_{y : F_j(y) \subset Q_n},\]
\[ G_n = \{ x : q(x, X - G) > 1/n \} \quad \text{and} \quad Q_n = \{ x : q(x, K) \leq 1/n \}, \]

and thus
\[ G = G_1 \cup G_2 \cup \ldots, \quad G_1 \subseteq G_2 \subseteq \ldots \]

and
\[ K = Q_1 \cap Q_2 \cap \ldots, \quad Q_1 \supset Q_2 \supset \ldots. \]

Proof of (12). 1. Let \( F(y) \subseteq G \). Since \( F(y) \) is compact, there is by (14) an \( n \) such that \( F(y) \subseteq G_n \). We have to show that there is \( j > m_n \) such that \( F_j(y) \subseteq G_n \). Suppose that the contrary is true, i.e., that \( F_j(y) - G_n = \emptyset \) for each \( j > m_n \); but then
\[ \lim_{j \to \infty} F_j(y) - G_n = \emptyset, \]
which is a contradiction.

2. Let \( F(y) - G = \emptyset \). Let \( p \in F(y) - G \). By (11) there is, for each \( n \) and \( j > m_n \), a point \( p_j \in F_j(y) \) such that \( |p_j - p| \leq 1/n \), hence \( q(p_j, X - G) \leq 1/n \), i.e., \( p_j \notin G_n \). Thus \( F_j(y) \notin G_n \).

Proof of (13). 1. Let \( F(y) \subseteq K \). Suppose that contrary to (13), there are \( n \) and \( j > m_n \) such that \( F_j(y) \notin Q_n \). Let \( q_j \in F_j(y) - Q_n \), i.e., \( q(q_j, K) > 1/n \) and consequently \( q(q_j, F(y)) > 1/n \). Therefore \( \text{dist}(F_j(y), F(y)) > 1/n \), contrary to (11).

2. Let \( F(y) - K = \emptyset \). Let \( p \in F(y) - K \). Therefore there is \( n \) such that \( q(p, K) > 1/n \), i.e. \( p \notin Q_n \). Suppose that for each \( j > m_n \), we have \( F_j(y) \subseteq Q_n \). Then
\[ \lim_{j \to \infty} F_j(y) = Q_n, \]
which is a contradiction.

Remark. A similar formula to (13) is known for point-valued mappings (see [4], p. 268, and [8], p. 386).

Theorem 6. Let \( X_n \) be compact and let \( X = X_0 \times X_1 \times \ldots \). If each mapping \( F_n : Y \to 2^{X_n} \) belongs to the class \( L^+ \) (respectively to the class \( L_- \)) for \( n = 0, 1, \ldots \), then so does their Cartesian product \( F = F_0 \times F_1 \times \ldots \).

The proof is completely analogous to the proof of Theorem 12 of [9].

7. Selection problems. In this section we assume that \( L \) is a \( \sigma \)-algebra and that \( X \) is a Polish (complete separable) space (\( 2^X \) is endowed with the Vietoris topology).

According to Theorem and Corollary 1 of [10], the following Selection Theorem is true.

Theorem. If \( F : Y \to 2^X \) is either \( L^+ \) or \( L_- \), there exists a selector \( f : Y \to X \) (i.e., \( f(y) \in F(y) \)) which is an \( L \)-mapping.
Corollary. For each $L$-measurable mapping $F : Y \to 2^X$ there exists an $L$-measurable selector $f : Y \to X$.

For further applications (also to the optimal control theory) see e.g. [5], [6], [7], and [12].

III. Case where $Y$ is a topological space

8. Relations to continuous, closed, open mapping etc.

The proof of the following theorem is immediate.

Theorem 1. Let us assume that all open subsets of $Y$ are members of the lattice $L$. Then each continuous mapping $F : Y \to \mathcal{C}(X)$ is an $L$-mapping.

More precisely: if $F$ is upper (lower) semi-continuous, then $F \in L^+$ ($F \in L_-$).

Corollary. If $Y = \mathcal{C}(X)$, then the identity, $F(K) = K$, is an $L$-mapping.

Theorem 2. Let $f : X \to Y$ be continuous and onto and let $f^{-1}(y)$ be compact for each $y \in Y$. Then the mapping $f^{-1} : Y \to \mathcal{C}(X)$ satisfies the equivalences:

$$(f^{-1} \in L^+) \equiv (f(K) \in (-L) \text{ for each closed } K \text{ in } X),$$
$$(f^{-1} \in L_-) \equiv (f(G) \in L \text{ for each open } G \text{ in } X).$$

This follows by virtue of the general valid formula (see [8], p. 14(3)):

$$f(A) = \{y : A \cap f^{-1}(y) \neq \emptyset\}$$

in which one has to substitute for $A$ either $K$ or $G$ (compare (4) and (6)).

Remark. In the case where $L$ is the lattice of all open subsets of $Y$, our theorem states that $f^{-1}$ is upper semi-continuous iff $f$ is a closed mapping; $f^{-1}$ is lower semi-continuous iff $f$ is an open mapping (compare [8], p. 177).

9. The graph of the relation $x \in F(y)$. The set

$$J = \{\langle x, y \rangle : x \in F(y)\}$$

is the graph under consideration.

Theorem. Let $M$ be a lattice of subsets of $X \times Y$, closed under countable unions and such that

$$(G \text{ open in } X \text{ and } A \in L) \Rightarrow (G \times A) \in M.$$ \hspace{1cm}

Let $F \in L^+$, then $[(X \times Y) - J] \in M$. 

Proof. Let $G_1, G_2, \ldots$ be an open base of $X$. Then: $x \notin F(y)$ iff there is $n$ such that $x \in G_n$ and $F(y) \cap \bar{G}_n \neq \emptyset$, i.e.,

$$(X \times Y) - J = \bigcup_n \{ \langle x, y \rangle : (x \in G_n) \land (F(y) \cap \bar{G}_n = \emptyset) \} = \bigcup_n [G_n \times \{ y : F(y) \cap \bar{G}_n = \emptyset \}].$$

Since $F \in L^+$, we have, by (4), $\{ y : F(y) \cap \bar{G}_n = \emptyset \} \in L$; this completes the proof.

**Corollary 1.** If $F$ is upper semi-continuous, then $J$ is closed in $X \times Y$.

Here $L$ denotes the lattice of all open subsets of $Y$.

**Corollary 2.** If $F$ is of Baire class $\alpha^+$, then $J$ is of Borel multiplicative class $\alpha$ in $X \times Y$.

Here $L$ denotes the lattice of Borel subsets of additive class $\alpha$ of $Y$.

**Corollary 3.** If $L$ is any projective class (in $Y$), then $J$ belongs to the projective class $(-L)$ in $X \times Y$.

**Corollary 4.** If $F$ is $L$-measurable, so is $J$.

For the sake of simplicity we put here $X = Y = \text{interval}$.

Remark. In the case where $F(y)$ reduces to a single point, $f(y)$, our Theorem implies some well known statements about the graph of the mapping $f$

$$J_0 = \{ \langle x, y \rangle : x = f(y) \}$$

(see e.g. [8], p. 384 and [3], Section 3).

Let us note that the converse to Corollary 2 is not true: the set $J_0$ can be $G_\delta$ without $f$ being of class $1$.

**IV. Final remarks**

It seems useful in many cases to consider the set $\Phi$ of all mappings $F : Y \to \mathcal{C}(X)$ as a metric space. The distance between two members $F_0$ and $F_1$ of $\Phi$ is defined following the regular procedure (compare [8], p. 218). Namely — assuming that $X$ is bounded — we put

$$\text{Dist} (F_0, F_1) = \sup \text{dist} [F_0(y), F_1(y)] \quad \text{where} \quad y \in Y$$

and where “dist” means the Hausdorff distance of sets.
In view of this definition, convergence in the space $\Phi$ means the uniform convergence. Thus Theorem 5 of § 6 can be restated as follows.

*The sets $L^+$ and $L_-$ are closed in the space $\Phi$.***

The space $\Phi$ — although not separable, in general — has a number of interesting properties. For example, if $X$ is complete, then so is $\ell(X)$ and consequently (see [8], p. 408) also $\Phi$ is complete.

Furthermore, the set of semi-continuous compact-valued mappings can be shown to be non-dense in the set of Baire 1st class mappings (under suitable assumptions on $X$ and $Y$).

An analogous statement is true also for Baire mappings of arbitrary class $\alpha$.

Let us add that “joint semi-continuity” holds under the above defined topology of the space $\Phi$.

The proofs of these statements and of further properties of the space $\Phi$ will appear elsewhere.

References