Let $X$ be a nonempty set, $2^X$ the algebra of all subsets of the set $X$ and $\lambda$ the convergence closure operator on $2^X$. We recall its definition. For each $\mathcal{M} \subset 2^X$,

$$\lambda \mathcal{M} = \{A; A \in 2^X \text{ and there is a sequence of sets } A_n \in \mathcal{M} \text{ such that} \}$$

$$A = \lim_{n \to \infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n.$$ 

A power of $\lambda$ is defined by the transfinite induction: $\lambda^0 \mathcal{M} = \mathcal{M}$, $\lambda^\alpha \mathcal{M} = \bigcup_{\beta < \alpha} \lambda(\lambda^\beta \mathcal{M})$ for an ordinal $\alpha \neq 0$ and $\mathcal{M} \subset 2^X$.

Let a set algebra $\mathcal{A}$, $\mathcal{A} \subset 2^X$, be given. It has been noticed (see [2]) that $\lambda^\alpha \mathcal{A}$ is also a set algebra for an arbitrary ordinal $\alpha$ and $\lambda^{\omega_1} \mathcal{A}$ ($\omega_1$ = the first uncountable ordinal) is equal to the $\sigma$-algebra generated by $\mathcal{A}$. An easy completion of the well-known statement (see [1], Chap. 1, Exercise 13) claims:

(1) The image $P[\lambda^{\omega_1} \mathcal{A}] = \{PA; A \in \lambda^{\omega_1} \mathcal{A}\}$ is a closed subset of the real line for each probability measure $P$.

J. Novák has raised the problem to find the least ordinal $\alpha$ such that $P[\lambda^\alpha \mathcal{A}]$ is always closed. The answer is given by

**Theorem.** The number $\alpha = 2$ is the least ordinal such that $P[\lambda^\alpha \mathcal{A}]$ is closed.

**Proof.** 1) $P[\lambda^2 \mathcal{A}]$ is closed. Let a real number $a$ belong to the closure of $P[\lambda^2 \mathcal{A}]$. From (1) it follows that there is $B \in \lambda^{\omega_1} \mathcal{A}$ such that $PB = a$. The definition of the outer measure implies the existence of sets $B_n \in \mathcal{A}$, $n = 1, 2, \ldots$, $i = 1, 2, \ldots$, such that $a \leq P(\bigcup B_n) \leq a + 1/n$ and $B \subset \bigcup_{i=1}^{\infty} B_n \in \lambda \mathcal{A}$ for each $n = 1, 2, \ldots$. Hence

$$a = P(\bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_n) \in P[\lambda^2 \mathcal{A}].$$

2) The image $P[\lambda \mathcal{A}]$ need not be closed as the following example shows.

Let $R$ denote the real line, $\mathcal{A}$ the algebra generated by semiclosed intervals of the form $\langle a, b \rangle$, $a, b \in R$. 

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Lemma. If \( A_n \in \mathcal{R}, \ n = 1, 2, \ldots, A = \lim_{n \to \infty} A_n \), then there is a set \( Y \subseteq A \) or \( Y \cap A = \emptyset \).

Proof. We denote \( B_i = R + A_i + A_{i+1} \), \( C_i = \bigcap_{n=i}^{\infty} B_n \), \( i = 1, 2, \ldots \), where \( + \) denotes the symmetric difference. Evidently \( B_i \in \mathcal{R} \) and

\[
\bigcup_{n=1}^{\infty} C_n = R.
\]

Now two cases are possible:

1) There is a natural number \( n_0 \) such that \( C_{n_0} = R \). Then \( B_{n_0} = B_{n_0+1} = \ldots = R \) and \( A_{n_0} = A_{n_0+1} = \ldots = A \). At least one of the sets \( Y = A_{n_0} \) or \( Y = R - A_{n_0} \) possesses the declared property.

2) All the sets \( C_n \neq R \). Then there is \( B_k \neq R \). We choose a compact non-degenerate interval \( T_1 \subset R - B_k \). Suppose that compact non-degenerate intervals \( T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n \) have been constructed in such a way that \( T_i \cap C_i = \emptyset \), \( i = 1, 2, \ldots, n \). Denote \( T_n^* = T_n - \{r\} \), where \( r \) is the right end point of \( T_n \). Now two cases are possible:

a) \( T_n^* \subset C_{n+1} \). Then \( A_{n+1} \cap T_n^* = A_{n+2} \cap T_n^* = \ldots = A \cap T_n^* \) (otherwise there would be \( k > n \) and a point \( x \in \left(A_k + A_{k+1}\right) \cap T_n^* \subset (R - B_k) \cap C_{n+1} = \emptyset \)). Hence the set \( Y \) from the Lemma can be found by using a nonempty measurable subset of \( T_n^* \cap A \) or \( T_n^* - A \).

b) \( T_n^* \notin C_{n+1} \). Then there is a point \( x \in T_n^* \), \( x \notin C_{n+1} \), i.e. there is \( j \geq n + 1 \) such that \( x \notin B_j \). We pick out a non-degenerate compact interval \( T_{n+1} \subset T_n^* - B_j \).

In the case a) we have the set \( Y \) as desired in the Lemma. If the case a) does not occur, then we have a non-increasing sequence of non-degenerate compact intervals \( T_n \). The intersection of it is disjoint with \( \bigcup_{n=1}^{\infty} C_n \) and nonempty. We get a contradiction with (2).

Example. Let \( Q = \{q_1, q_2, \ldots\} \) be the set of all rational numbers, \( s_n = q_n + \sqrt{2}, S = \{s_1, s_2, \ldots\} \). A probability \( P \) on \( \lambda^2 \mathcal{R} \) is defined by the relations \( P(\{q_n\}) = 2/3^{2n-1}, P(\{s_n\}) = 2/3^{2n}, P(R - Q - S) = 0 \). It is easy to see that sets \( A_n \in \mathcal{R} \) can be chosen in such a way that \( q_i \notin A_n \) and \( s_i \notin A_n \) for \( i = 1, 2, \ldots, n \). Then \( (3/4)(1 - 1/9^n) = \sum_{i=1}^{n} Pq_i \leq PA_n \leq 1 - \sum_{i=1}^{n} Ps_i = 3/4 + 1/(4 \cdot 9^n) \) and hence

\[
\lim_{n \to \infty} PA_n = 3/4.
\]

Now, let \( A \) be any set of \( \lambda^2 \mathcal{R} \) such that \( PA = 3/4 \). Then the uniqueness of the ternary expansion \( 3/4 = \sum_{i=1}^{\infty} 2/3^{2i-1} = \sum_{i=1}^{\infty} Pq_i = P(Q) \) implies \( Q \subset A \) and \( A \cap S = \emptyset \). Then \( A \notin \lambda \mathcal{R} \) as follows from the Lemma and from the fact that \( Q \) and \( S \) are dense subsets of \( R \). It follows that \( 3/4 \) is a point of the closure of the P-image of \( \mathcal{R} \) but there is no element \( A \in \lambda \mathcal{R} \) such that \( PA = 3/4 \).
Remark. Part 1) of the proof of the Theorem can be proved without using an outer measure. The outer measure can be replaced by Marczewski’s characteristic function of a sequence of sets (see [3]). Problems and importance of elimination of the notion of an outer measure from measure theory are treated in [2].

References


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